MATH-3

COMPLEX NUMBERS

CHAPTER - 1

1.1 Introduction :

We have the knowledge of Natural numbers (N), Integers (I or Z), Rationals (Q) and irrational numbers (Q'). All these numbers constitute real number (R). One of the properties of real numbers is square of every real number is positive i.e $x^2 \geq 0$ for every real x. i.e. there is no real number whose square is negative.

The equation $x_2 + 1 = 0$ have no solution in real numbers. So we have to extend a new kind of numbers. We define the square root of a negative number as imaginary number.

Particularly $\sqrt{-1} = i$

Positive integral powers of i :-

We have $i = \sqrt{-1}$

$i_2 = -1$

$i_3 = -i$

$i_4 = (i_2)_2 = (-1)_2 = 1$

In order to compute i_n for n > 4, we divide n by 4 and obtain the remainder r. Let m be the quotient. When n is divided by 4. Then

n = 4m + r,

$i_n = i_{4m+r} = (i_4)_m \cdot i_r = 1.i_r = i_r$

Thus the value of i_n for n>4 is i_r, where r is the reminder when n is divided by 4.

Ex : $i_{135} = i_3 = -i$

$$\begin{array}{r|l|l} 4 & 135 & 33 \\ & 12 & \end{array}$$

$$\begin{array}{r} 15 \\ 12 \\ \hline 3 = r \end{array}$$

1.2 Complex Number

The number of the form a + ib is called a Complex Number, where ‘a’ and ‘b’ are real numbers and $i = \sqrt{-1}$ is an imaginary number.

In complex number z = a + ib, the numbers a and b are respectively known as real and imaginary parts of z and we write :

Re (Z) = a and Im (Z) = b

Thus the set C of all complex numbers is given by

C = {z : z = a + ib, where a, b $\in$ R}

Example : 2 + 3i, $\sqrt{2} + 3i$, 5 - 4i etc.

Purely real and purely imaginary number :

A complex number z = a + bi is said to be

i) Purely real, if Im (z) = 0 i.e. b = 0

ii) Purely imaginary, if Re(z) = 0 i.e a = 0

e. g. 2 = 2 + 0.i, -1 = -1 + 0.i are purely real.

3i = 0 + 3i, -i = 0 + (-1)i are purely imaginary

Hence every purely real number and every purely imaginary number are complex number.

Equality of complex number :

Two complex number $z_1 = a_1 + ib_1$, and $z_2 = a_2 + ib_2$

are equal if $a_1 = a_2$ and $b_1 = b_2$

i.e $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$

e.g If $z_1 = 2 - iy$, $z_2 = x + 3i$, $z_1 = z_2$, find x and y

Ans. $z_1 = z_2$

$\Rightarrow 2 - iy = x + 3i$

$\Rightarrow 2 = x$ and $y = -3$

Modulus of a complex number :

The modulus of a complex number z = a + ib is denoted by $|z|$ and is defined as

$$|z| = \sqrt{a^2 + b^2} = \sqrt{\{Re(z)^2\} + \{Im(z)^2\}}$$

Example : z = 3 + 4i

$$|z| = \sqrt{3^2 + 4^2} = 5$$

Note : In the set C of all complex number, the ordered relation is not defined. As such $z_1 > z_2$, $z_2 > z_1$ has no meaning but $|z_1| > |z_2|$ or $|z_1| < |z_2|$ has got its meaning. Since $|z_1|$ and $|z_2|$ are real number.

Polar form of a Complex number :

Let $X'OX$ & YOY' be the co-ordinate axis. Let z = a + ib be represented by a point P(a, b). Draw $PM \perp OX$. Then OM =a and PM = b. Join OP. Let OP = r and $\angle XOP = \theta$

Then a = r cos θ and b =r sin θ

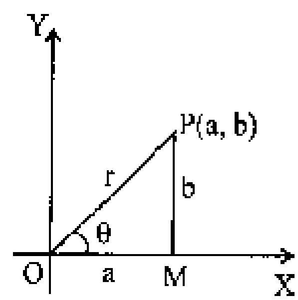
z = a+ ib = r (cos θ + i sin θ) is called the polar form of a complex number z.

Comparing real and imaginary parts, we get.

$a = r\cos\theta$ .... (1)

$b = r\sin\theta$ .... (2)

$\Rightarrow r = \sqrt{a^2 + b^2} = |z|$ (modulus of the complex number)



Dividing (2) by (1), we get

$$\tan\theta = \frac{b}{a}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

The angle is known as the amplitude or argument of z written as amp (z) or arg (z).

The unique value of θ such that - $\pi < \theta \leq \pi$ for which a = r cos and b = r sin is known as the principal value of the amplitude.

The general value of the amplitude is $(2n\pi + \theta)$, where n is an integer and is the principal value of amp (z). While reducing a complex number to polar form, we always take the principal value.

Simple process to find Arg z :

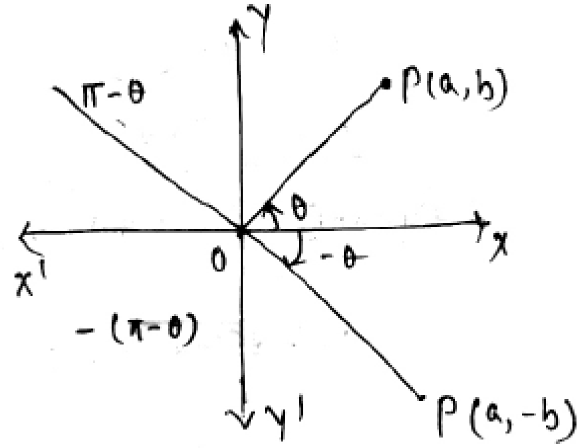
Let z = a+ib, = $\tan^{-1}\left|\frac{b}{a}\right|$

x > 0, y > 0 $\Rightarrow$ Pr arg z = θ

x <0, y > 0 $\Rightarrow$ Pr arg z = $\pi - \theta$

x >0, y < 0 $\Rightarrow$ Pr arg z = - θ

x <0, y < 0 $\Rightarrow$ Pr arg z = - (π-θ) or $\pi + \theta$



For Example : Prove that $\arg\left(\frac{-1+\sqrt{3}\,i}{2}\right) = \frac{2\pi}{3}$

Let z = $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$a = -\frac{1}{2},\ b = \frac{\sqrt{3}}{2}$

$\theta = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

Here a < 0, b > 0

$\Rightarrow$ θ lies in 2nd quadrant

$\therefore$ arg z = $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$

1.2 Algebra of Complex Numbers

i) Addition of two complex numbers :

Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$ be two complex numbers, then their sum is $z_1 + z_2$ is defined as the $(a_1 + a_2) + i(b_1 + b_2)$

e.g $z_1 = 2 + 3i$, $z_2 = 3 + 4i$

= (2+3) + (3+4)i = 5 + 7i

ii) Subtraction of two complex numbers :

If $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$ be two complex number then their difference is $z_1 - z_2$ is defined as $(a_1 - a_2) + i(b_1 - b_2)$

e.g. $z_1 = 2 + 3i$, $z_2 = 4 + \sqrt{3}\,i$

$z_1 - z_2 = (2-4) + (3-\sqrt{3})i = -2 + (3-\sqrt{3})i$

iii) Multiplication of two complex numbers

Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$ be two complex numbers. Then the multiplication of z_1 with z_2 is denoted by z_1z_2 and is denoted as the complex number $(a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$

e.g.

Let $z_1 = 2 + 4i$ and $z_2 = 4 + 5i$

$z_1z_2 = (2+4i)(4+5i) = (2.4 - 4.5) + i(2.5 + 4.4)$

= (8 - 20) + i (10 + 16)

= -12 + 26i

Conjugate of a complex number :

Let z = a + ib be a complex number. Then the conjugate of z is denoted by $\bar{z}$ and is equal to a - ib.

e.g. z = 2 + 3i, $\bar{z}$ = 2 - 3i

or The conjugate of a complex number 'z' denoted by $\bar{z}$, which is the complex number obtained changing the sign imaginary part of z.

e.g.

$$\overline{(2+3i)} = (2-3i)$$

$$\overline{(3+5i)} = (3-5i), \overline{6i} = -6i, \overline{-2i} = 2i \text{ etc.}$$

iv) Reciprocal of a complex number / Multiplicative inverse of a complex no :

Let Z = a+ib be a non zero complex number. Then the multiplicative inverse of z is same as its reciprocal, i.e $\frac{1}{z}$.

$$\frac{1}{z} = \frac{\text{Re } z}{|z|^2} + i\left(\frac{-\text{Im } z}{|z|^2}\right) = \frac{\bar{z}}{|z|^2}$$

Example : Find the multiplicative inverse of 2-3i.

Ans. Let z = 2-3i

$$\therefore \quad z^{-1} = \frac{1}{z} = \frac{1}{2-3i} = \frac{\overline{2-3i}}{|2-3i|^2} = \frac{2+3i}{2^2+(-3)^2} = \frac{2+3i}{4+9} = \frac{2+3i}{13} = \frac{2}{13} + i\frac{3}{13}$$

v) Division of two complex numbers :

The division of a complex number z_1 by a non-zero complex number z_2 is defined as the multiplication of z_1 with the multiplicative inverse z_2 and is denoted by $\frac{z_1}{z_2}$

$$\frac{z_1}{z_2} = z_1 . z_2^{-1} = z_1\left(\frac{1}{z_2}\right)$$

e.g.

z_1 = 2 + 3i, z_2 = 1 + 2i

$$= \frac{z_1}{z_2} = (2+3i).\left(\frac{1}{1+2i}\right) = (2+3i).\left(\frac{1-2i}{(1+2i)(1-2i)}\right)$$

$$= (2+3i)\left(\frac{1-2i}{1-(-4)}\right) = (2+3i)\left(\frac{1-2i}{5}\right) = \left(\frac{2}{5}+\frac{6}{5}\right) + i\left(\frac{-4}{5}+\frac{3}{5}\right) = \frac{8}{5} - \frac{1}{5}i$$

To express the given complex number in the standard form a+ib.

Algorithm :

Step : 1

Write the given complex number in the form $\frac{a+ib}{c+id}$ (by using fundamental operations of addition, subtraction and multiplication).

Step : 2

Multiply both the numerator and denominator by conjugate of the denominater, then simplify

Example : Express in the form of a+ib, $\frac{4-3i}{(1-i)^2}$

Ans. $\frac{4-3i}{(1-i)^2}=\frac{4-3i}{1+i^2-2i}=\frac{4-3i}{1-1-2i}=\frac{4-3i}{-2i}=\frac{(4-3i)}{(-2i)(2i)}=\frac{8i-6i^2}{-4i^2}=\frac{8i-6(-1)}{-4(-1)}$

$=\frac{6+8i}{4}=\frac{6}{4}+\frac{8}{4}i=\frac{3}{2}+2i$

1.4 <u>Properties of Modulus :</u>

If $z, z_1, z_2 \in$ C then

i) $|z|=0 \Leftrightarrow z=0$ i.e. $\text{Re}(z)=\text{Im}(z)=0$

ii) $|z|=|\bar{z}|=|-z|$

iii) $-|z|\le \text{Re}(z)\le |z|;\ -|z|\le \text{Im}(z)\le |z|$

iv) $z.\bar{z}=|z|^2$

v) $|z_1z_2|=|z_1|.|z_2|$

vi) $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|},\ z_2\ne 0$

vii) $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2\text{Re}(z_1.\bar{z}_2)$

viii) $|z_1+z_2|^2=|z_1|^2+|z_2|^2-2\text{Re}(z_1.\bar{z}_2)$

ix) $|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$

Proof : (vii) $(z_1+z_2)^2=(z_1+z_2)(\overline{z_1+z_2})$

$=(z_1+z_2)(\bar{z}_1+\bar{z}_2)=z_1\bar{z}_1+z_2\bar{z}_1+z_1\bar{z}_2+z_2\bar{z}_2$

$=|z_1|^2+|z_2|^2+|z_1\bar{z}_2|+(\overline{z_1\bar{z}_2})\ (z+\bar{z}=2\text{Re}(z))$

$=|z_1|^2+|z_2|^2+2\text{Re}(z_1.\bar{z}_2)$

<u>Triangle Inequality :</u> (Inequality Involving Complex no.) : - For two complex number z_1 and z_2.

$|z_1+z_2|\le |z_1|+|z_2|$

Proof : $|z_1+z_2|^2=(z_1+z_2)(\overline{z_1+z_2})$

$=(z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$=z_1.\bar{z}_1+z_2.\bar{z}_1+z_1.\bar{z}_2+z_2.\bar{z}_2$

$=|z_1|^2+|z_2|^2+z_1.\bar{z}_2+(\overline{z_1.\bar{z}_2})$

$=|z_1{}^2|+|z_2{}^2|+2\text{Re}(z_1.\bar{z}_2)$

$2\text{Re}(z_1.\bar{z}_2)\le 2|z_1.\bar{z}_2|\quad (\ -|z|\le \text{Re}(z)\le |z|)$

$\le |z_1|^2+|z_2|^2+2|z_1.\bar{z}_2|^2$

$\le |z_1|^2+|z_2|^2+2|z_1|.|\bar{z}_2|\quad (\ |z_1z_2|=|z_1|.|z_2|)$

$\le |z_1|^2+|z_2|^2+2|z_1||z_2|\quad (\ |z|=|\bar{z}|)$

$\leq \left(|z_1| + |z_2|^2\right)$

$\Rightarrow |z_1 + z_2| \leq |z_1| + |z_2|$ (Proved)

Corollary : $|z_1 - z_2| \geq |z_1| - |z_2|$

Proof : $|z_1| = |(z_1 - z_2) + z_2|$

$\leq |z_1 - z_2| + |z_2|$

$\Rightarrow |z_1| - |z_2| \leq |z_1 - z_2|$

Square root of a complex number :

Let $\sqrt{a + ib} = x + iy \ (a, b, x, y \in R)$

Squaring both sides;

$a + ib = x_2 - y_2 + 2xyi$

Equating the real and imaginary parts from both sides,

$x_2 - y_2 = a$ ....(1)

and $2xy = b$

$(x_2 + y_2)_2 = (x_2 - y_2) + 4x_2y_2 = a_2 + b_2$

$\Rightarrow x_2 + y_2 = \pm\sqrt{a^2 + b^2}$

$\Rightarrow x_2 + y_2 = \sqrt{a^2 + b^2}$ ....(2)

(Rejecting the negative sign as $x_2 + y_2 > 0$)

Adding and subtracting equation (1) and (3) we get,

$x^2 = \frac{1}{2}\left(\sqrt{a^2 + b^2} + a\right)$ and $y^2 = \frac{1}{2}\left(\sqrt{a^2 + b^2} - a\right)$

$\therefore x = \pm\left[\frac{1}{2}\left(\sqrt{a^2 + b^2} + a\right)\right]^{\frac{1}{2}}$ and $y = \pm\left[\frac{1}{2}\left(\sqrt{a^2 + b^2} - a\right)\right]^{\frac{1}{2}}$

<u>Note :</u> Since $2xy = b$, xy and b have the same sign. In other words, if b is positive then x and y are of the same sign, if b is negative then x and y are of opposite sign.

Algorithm for solving square root of a complex number a+ib.

Step - 1 Take $\sqrt{a + ib} = \pm(x + iy)$

Step - 2 Squaring both sides of step - 1

Step - 3 Comparing the real and imaginary part of step - 2

$x_2 - y_2 = a$ .... (1)

$2xy = b$ .... (2)

Step - 4 find $x_2 + y_2$

(using the formula $(a+b)_2 = (a-b)_2 + 4ab$)

$(x_2+y_2)_2 = (x_2 - y_2)_2 + 4x_2y_2$

Let $x_2 + y_2 = C$. (say) .... (3)

(here C > 0)

Step - 5 Solving equation (1) & (3) as equation(1) + Equation(3) to get the value of x.

Step - 6 Put the value of x in equation (2) to get the value of y.

Step - 7 Substitute the values of x & y in step (1) to get square root of a + ib.

Example : Obtain the square root of 3+4i

Solution : Let $\sqrt{3+4i} = \pm(x+iy)$, where $x, y \in R$ ....1)

$\Rightarrow 3 + 4i = x_2 - y_2 + 2xyi$

Equating the real and imaginary parts we get.

$x_2 - y_2 = 3$ .... 2)

$2xy = 4$ .... 3)

$\therefore x^2 + y^2 = \sqrt{(x^2 - y^2)^2 + (2xy)^2} = \sqrt{3^2 + 4^2} = 5$

$x_2 + y_2 = 5$ .... 4)

Adding equations (2) & (4) we get

$x_2 + y_2 = 5$

$x_2 - y_2 = 3$

$2x_2 = 8$

$\Rightarrow$ $x_2 = 4$

$\Rightarrow$ $x = 2$

Putting the value of x in Equation3, wet get

$2.2.y = 4$

$\Rightarrow y = 1$

Hence $\sqrt{3+4i} = \pm(2+i)$

or **Alternative Method**

For any complex number z = a+ib, we have the square root of z

i) $\sqrt{z} = \pm\left\{\sqrt{\frac{|z| + \text{Re } z}{2}} + i\sqrt{\frac{|z| - \text{Re } z}{2}}\right\}$, if Im(z)>0

ii) $\sqrt{z} = \pm\left\{\sqrt{\frac{|z| + \text{Re } z}{2}} - i\sqrt{\frac{|z| - \text{Re } z}{2}}\right\}$, if Im(z)<0

Here $|z| = \sqrt{a^2 + b^2}$, Re(z)=a, Im(z)=b

Algorithm :

Step : 1 Let Z=a+ib

Step : 2 Identify real part(a) and imaginary part (b) of Z.

Calculate modulus of z= |z| = $\sqrt{a^2+b^2}$

Step : 3 Put the values of |z| and Re Z in the following (according to the sign of Im Z

i) $\sqrt{z}=\pm\left\{\sqrt{\frac{|z|+\operatorname{Re} z}{2}}+i\sqrt{\frac{|z|-\operatorname{Re} z}{2}}\right\}$, if Im(z)>0

ii) $\sqrt{z}=\pm\left\{\sqrt{\frac{|z|+\operatorname{Re} z}{2}}-i\sqrt{\frac{|z|-\operatorname{Re} z}{2}}\right\}$, if Im(z)<0

Example

Find the square root of -8+i

Solution

Let z = -8+i

Here Re(z)=-8, Im(z)=1 >0

|z| = $\sqrt{(-8)^2+1^2}=\sqrt{65}$

= $\pm\left\{\sqrt{\frac{\sqrt{65}-8}{2}}+i\sqrt{\frac{\sqrt{65}+8}{2}}\right\}$

Example :

Find the square root of 5-12i

Solution : Let z = 5-12i

Here Re(z)=5, Im(z) = -12<0

$|z|=\sqrt{5^2+(-12)^2}=\sqrt{25+144}=\sqrt{169}=13$

∴ $\sqrt{z}=\pm\left\{\sqrt{\frac{|z|+\operatorname{Re} z}{2}}-i\sqrt{\frac{|z|-\operatorname{Re} z}{2}}\right\}$

= $\pm\left\{\sqrt{\frac{13+5}{2}}-i\sqrt{\frac{13-5}{2}}\right\}$ = $\pm\left\{\sqrt{9}-i\sqrt{4}\right\}$ = $\pm\{3-2i\}$

1.5 <u>Cube roots of unity :</u>

Let $x=\sqrt[3]{1}$

x_3 = 1

x_3 - 1 = 0

(x - 1) (x_2 + x + 1) = 0

(x - 1) = 0 or (x_2 + x + 1) = 0

x = 1 or $x=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{3}\,i}{2}$

∴ The cube roots of unity are 1, $\frac{-1+\sqrt{3}\,i}{2}$ and $\frac{-1-\sqrt{3}\,i}{2}$

Let $\frac{-1+\sqrt{3}i}{2}$ be denoted by ω (womega)

$$\therefore\left(\frac{-1+\sqrt{3}\,i}{2}\right)^2=\frac{-1-3-2i\sqrt{3}}{4}=\frac{-2-2i\sqrt{3}}{4}=\frac{-1-1i\sqrt{3}}{2}$$

Hence $\omega^2=\dfrac{-1-\sqrt{3}i}{2}$

Also, since ω is a cube root of unity, ω_3 = 1

Properties of cube roots of Unity :

If $\sqrt[3]{1}=1,\omega,\ \omega^2$ where $\omega=\dfrac{-1+\sqrt{3}\,i}{2}$ and $\omega^2=\dfrac{-1-\sqrt{3}\,i}{2}$

then we notice the following properties.

(1) The sum of cube roots of unity is zero.

i.e. 1 + w + w_2 = 0

Proof : 1 + w + w_2 = $1+\left(\dfrac{-1}{2}+\dfrac{\sqrt{3}}{2}i\right)+\left(\dfrac{-1}{2}-\dfrac{\sqrt{3}}{2}i\right)$ = 1 + (-1) = 0

1 + w + w_2 = 0

(2) Cubes of cube root of unity.

Proof : We have ω_3 = 1

$$\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)^3=\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)^2\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)=\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$=\left(\frac{-1}{2}-i\frac{\sqrt{3}}{2}\right)^2=\frac{1}{4}+\frac{3}{4}=1$$

(3) Each complex cube root of unity is the square of the other.

Proof : $\left(\dfrac{-1}{2}+i\dfrac{\sqrt{3}}{2}\right)^2=\dfrac{1}{4}\left(1-2\sqrt{3}i-3\right)$

$$=\frac{1}{4}\left(-2-2\sqrt{3}i\right)=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

Similarly

(4) Product of three cube roots of unity is 1.

Proof : $1.\left(-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right)\left(-\dfrac{1}{2}-i\dfrac{\sqrt{3}}{2}\right)$

$$=\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2=\frac{1}{4}+\frac{3}{4}=1$$

(5) The complex roots ω and ω_2 are conjugate each other.

Proof : $\overline{\omega}=\left(\overline{-\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}}\right)=-\dfrac{1}{2}-i\dfrac{\sqrt{3}}{2}=\omega^2$

Similarly $\overline{\omega^2} = \omega$

(6) Each complex cube roots of unity is the reciprocal to the other.

Proof : The complex cube roots of unit are : ω, ω^2

Since $\omega . \omega^2 = \omega^3 = 1$

$$\omega = \frac{1}{\omega^2} \text{ and } \omega^2 = \frac{1}{\omega}$$

<u>Note :</u> (i) If ω be a cube root of unity and n be a positive integer, then $\omega^n = \omega^r$, when r is the least non-negative reminder by dividing n by 3.

$$\omega^n = \omega^{3x+r} = \omega^{3x}.\omega^r = (\omega^3)^x.\omega^r = 1.\omega^r = \omega^r \text{, r = 1 or 2}$$

Thus $\omega^7 = (\omega^3)^2.\ \omega = 1.\ \omega = \omega$

$\omega^{29} = (\omega^3)^9.\ \omega^2 = 1.\ \omega^2 = \omega^2$

$\omega^{42} = (\omega^3)^{14} = 1$

(ii) $1 + \omega + \omega^2 = 0$

$\Rightarrow 1 + \omega = -\omega^2$ and $1 + \omega^2 = -\omega$

Procedure for solving the problems containing ω^n

Step :1

Covert ω^n (n>3) into either I, ω or ω^2,

by $\omega^n = \omega^{3x} = 1$, or $\omega^n = \omega^{3x+1} = \omega$, or $\omega^n = \omega^{3x+2} = \omega^2$

Step : 2

Simplify by using the properties

$1 + \omega + \omega^2 = 0$, or $1 + \omega = -\omega^2$, or $1 + \omega^2 = -\omega$, or $\omega + \omega^2 = -1$ and $\omega^3 = 1$

Example :

Prove that $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11}) = 49$

Proof :

L.H.S. $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11})$

$= (2-\omega)(2-\omega^2)(2-\omega^{3.\,3+1})(2-\omega^{3.3+2})$

$= (2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2)$

$= (2-\omega)^2 (2-\omega^2)^2$

$= [(2-\omega)(2-\omega^2)]^2$

$= [4-2\omega^2-2\omega+\omega^3]^2$

$= [4-2\omega^2-2\omega+1]^2$

$= [5-2\omega-2\omega^2]^2$

$= [5-2(\omega+\omega^2)]^2$

$= [5-2(-1)]^2 = [5+2]^2 = [7]^2 = 49 =$ R. H. S.

<u>1.5 De-Moivre's theorem</u>

If n is an integer, positive or negative or zero.

Then $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$

<u>Proof : Case-I</u> : When n is a positive integer :

The proof is by mathematical induction.

Let P(n) : $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

If n = 1, P(1) is true.

Since $(\cos\theta + i\sin\theta)^1 = \cos(1.\theta) + i\sin(1.\theta)$

i.e. $\cos\theta + i\sin\theta = \cos\theta + i\sin\theta$

Assume that P(n) is true for some positive integer k

i.e. $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Now, $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k.(\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) = (\cos k\theta.\cos\theta - \sin k\theta.\sin\theta) + i(\sin\theta.\cos k\theta + \cos\theta.\sin k\theta) = \cos(k\theta+\theta) + i\sin(k\theta+\theta) = \cos(k+1)\theta + i\sin(k+1)\theta$

P(n) is true for n = k + 1, whenever P(k) is true.

By the principle of mathematical induction, P(n) is true for every positive integer n

i.e. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Case-II :

When n is a negative integer :

Let n = -m, where m is a positive integer.

$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \dfrac{1}{(\cos\theta + i\sin\theta)^m} = \dfrac{1}{\cos m\theta + i\sin m\theta}$ (by case-I)

$= \dfrac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)} = \dfrac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \cos m\theta - i\sin m\theta$

$= \cos(-m)\theta + i\sin(-m)\theta = \cos n\theta + i\sin n\theta$ ($\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$)

Case-III :

When n = 0

Clearly, $(\cos 0 + i\sin 0) = 1$

$= \cos(0.\theta) + i\sin(0.\theta)$

Thus, De Moivre's theorem is true for all integral value of n.

n_{th} Roots of Unity

Let $z = r(\cos\theta + i\sin\theta)$.

$z = 1 = \cos 0 + i\sin 0$

Then we may write,

$z = [\cos(2k\pi) + i\sin(2k\pi)]$, where k is a whole number.

$\therefore z^{\frac{1}{n}} = [\cos(2k\pi) + i\sin(2k\pi)]^{\frac{1}{n}}$; Where k = 0, 1, 2, ... (n - 1)

$= \cos\left(\dfrac{2k\pi}{n}\right) + i\sin\left(\dfrac{2k\pi}{n}\right)$..................(1) [By Demoivre's theorem]

Thus, we obtain nth root of unity

Example : Solve $z^4 = 1$

Ans. Let $z^4 = 1$

$\Rightarrow z = 1^{\frac{1}{4}}$

We know that $1 = \cos 0 + i\sin 0 = [\cos(2k\pi) + i\sin(2k\pi)]$

$$\therefore 1^{1/4} = \cos\left(\frac{2k\pi}{4}\right) + i\sin\left(\frac{2k\pi}{4}\right)$$

Where k = 0, 1, 2,3

$$\Rightarrow 1^{1/4} = \cos\left(\frac{k\pi}{2}\right) + i\sin\left(\frac{k\pi}{2}\right)$$

for k = 0, $1^{\frac{1}{4}}$ = cos0 + i sin0

for k = 1, $1^{1/4} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$

for k = 2, $1^{\frac{1}{4}}$ =cos π+ i sin π

for k = 3, $1^{1/4} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)$

I. Square roots of a complex number

For n = 2, putting k = 0, 1 in equation (i), we get ; For k = 0, $z^{1/2} = \sqrt{z} = r^{1/2}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]$

k = 1, $\sqrt{z} = r^{1/2}\left[\cos\frac{2\pi+\theta}{2} + i\sin\frac{2\pi+\theta}{2}\right]$

II. Square root of unity :

Since 1 = (cos θ + sin θ). We have

$1^{\frac{1}{2}} = \left[\cos\frac{(2k\pi+0)}{2} + i\sin\frac{(2k\pi+0)}{2}\right]$, where k = 0.1

$\therefore 1^{\frac{1}{2}} = (\cos k\pi + i\sin k\pi)$, where $k = 0,1$

Putting k = 0, 1, we obtain the square root of 1 as
(cos 0 + i sin 0) = 1 and (cos π + i sin π) = -1

Hence the square roots of 1 and -1.

Cube Roots of unity :

Since 1 = (cos 0 + i sin 0), we have

$1^{\frac{1}{3}} = \left[\cos\left(\frac{2k\pi+0}{3}\right) + i\sin\left(\frac{2k\pi+0}{3}\right)\right]$, Where k = 0, 1, 2

$1 = \left[\cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right)\right]$ .... (1) Where k = 0,1,2

Putting k = 0, in equation (1), we get.

cos 0 + i sin 0 = 1

Putting k = 1, in equation (1) we get

$$\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}\,i}{2}$$

Putting k = 2, in equation (1), we get., $\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \frac{-1-\sqrt{3}\,i}{2}$

Hence the cube roots of 1 are 1, $\frac{-1+\sqrt{3}\,i}{2}, \frac{-1-\sqrt{3}\,i}{2}$

Important Long type questions & Answers :

Q1. **If 1, ω, ω^2 are three cube roots of unity, Prove that**

$$\left(1-\omega+\omega^2\right)\left(1-\omega^2+\omega^4\right)\left(1-\omega^4+\omega^2\right) \quad \text{to2n factors} = 2^{2n}$$

Proof

L.H.S. $\left(1-\omega+\omega^2\right)\left(1-\omega^2+\omega^4\right)\left(1-\omega^4+\omega^2\right) \quad \text{to2n factors}$

$= \left(1-\omega+\omega^2\right)\left(1-\omega^2+\omega^{3+1}\right)\left(1-\omega^{3+1}+\omega^2\right) \quad \text{to2n factors}$

$= \left\{\left(1-\omega+\omega^2\right)\left(1+\omega-\omega^2\right)\right\}\left\{\left(1-\omega+\omega^2\right)\left(1+\omega-\omega^2\right)\right\} \quad \text{to n factors}$

$= \left\{\left(1-\omega+\omega^2\right)\left(1+\omega-\omega^2\right)\right\}\left\{\left(1-\omega+\omega^2\right)\left(1+\omega-\omega^2\right)\right\} \quad \text{to n factors}$

$= \left\{\left(1+\omega^2-\omega\right)\left(-\omega^2-\omega^2\right)\right\}\left\{\left(1+\omega^2-\omega\right)\left(-\omega^2-\omega^2\right)\right\} \quad \text{to n factors}$

$= \left\{\left(-\omega-\omega\right)\left(-2\omega^2\right)\right\}\left\{\left(-\omega-\omega\right)\left(-2\omega^2\right)\right\} \quad \text{to n factors}$

$= \left\{\left(-2\omega\right)\left(-2\omega^2\right)\right\}\left\{\left(-2\omega\right)\left(-2\omega^2\right)\right\} \quad \text{to n factors}$

$= \left\{4\omega^3\right\}\left\{4\omega^3\right\} \quad \text{to n factors}$

= 4.4......n times

= 4_n

= $(2_2)_n = 2_{2n}$ = RHS

Q2. Find the square root of 5 +12i

Solution : Obtain the square root of 3+4i

<u>**Solution :**</u> Let $\sqrt{5+12i} = \pm(x+iy)$, where $x, y \in R$ ....1)

$\Rightarrow$ 5 + 12i = $(x_2 - y_2)$ + 2xyi

Equating the real and imaginary parts we get.

$x_2 - y_2 = 5$ .... 2)

2xy = 12 .... 3)

$\therefore x^2 + y^2 = \sqrt{\left(x^2-y^2\right)^2 + (2xy)^2} = \sqrt{5^2+12^2} = 13$

$x_2 + y_2 = 13$ .... 4)

Adding equations 2 & 4 we get

$x_2 + y_2 = 13$

$x_2 - y_2 = 5$

$2x_2 = 18$

$\Rightarrow$ $x_2 = 9$

$\Rightarrow$ x = 3

Putting the value of x in Equation3, wet get

2.3.y = 12

$\Rightarrow$ y =2

Hence $\sqrt{5+12} = \pm(3+2i)$ **Ans.**

<u>**Alternatively**</u>

Let z = 5 + 12i

Here Re(z)=5, Im(z) = 12 > 0

$|z| = \sqrt{5^2 + 12^2} = \sqrt{25+144} = \sqrt{169} = 13$

$\therefore \quad \sqrt{z} = \pm\left\{\sqrt{\frac{|z|+\text{Re}\,z}{2}} + i\sqrt{\frac{|z|-\text{Re}\,z}{2}}\right\}$

$= \pm\left\{\sqrt{\frac{13+5}{2}} + i\sqrt{\frac{13-5}{2}}\right\} = \pm\left\{\sqrt{9} + i\sqrt{4}\right\} = \pm\{3+2i\}$

Q3. Find the square root of 8-5i

Solution : Let z = 8 - 5i

Here Re(z)=8, Im(z) = -5<0

$|z| = \sqrt{8^2 + (-5)^2} = \sqrt{89}$

$\therefore \quad \sqrt{z} = \pm\left\{\sqrt{\frac{|z|+\text{Re}\,z}{2}} - i\sqrt{\frac{|z|-\text{Re}\,z}{2}}\right\}$

$= \pm\left\{\sqrt{\frac{\sqrt{89}+8}{2}} - i\sqrt{\frac{\sqrt{89}-8}{2}}\right\} = \pm\left\{\sqrt{9} - i\sqrt{4}\right\} = \pm\{3-2i\}$

Q4. If $x + \frac{1}{x} = 2\cos\theta,$ **show that** $x^n + \frac{1}{x^n} = 2\cos n\theta$ **.**

Proof Let $x + \frac{1}{x} = 2\cos\theta$

$\Rightarrow \quad \frac{x^2+1}{x} = 2\cos\theta$

$\Rightarrow$ x_2+1=2xcos

$\Rightarrow$ x_2+$\cos_2$+$\sin_2$ = 2xcos

$\Rightarrow$ x_2+$\cos_2$-2xcos=-$\sin_2$ =-1.$\sin_2$=i_2 . $\sin_2$

$\Rightarrow$ (x-cos)$_2$ = (i sin)$_2$

$\Rightarrow$ x-cos=i sin

$\Rightarrow$ x = cos i sin

$\Rightarrow$ x = cos +i sin, cos - i sin

Let's choose x = cos +i sin

x_n = (cos +i sin)$_n$

$\Rightarrow$ x_n = cosn+i sinn

Again $\frac{1}{x^n} = x^{-n}$ = (cos +i sin)$_{-n}$

= cos(-n)+i sin(-n)

$\Rightarrow \quad \frac{1}{x^n} = x^{-n}$ =cosn- i sinn

Now $x^n + \frac{1}{x^n}$ = (cosn+i sinn) + (cosn-i sinn)

= cosn+i sinn+cosn-i sinn

= 2 cosn **(Proved)**

Q5. If cos α + cosβ+ cosγ = sinα+sinβ+ sinγ = 0, Then show that

cos 3α+cos3β + cos 3γ = 3. cos (α+β+γ)

Proof

Let $x = \cos\alpha + i \sin\alpha$

$y = \cos\beta + i \sin\beta$

$z = \cos\gamma + i \sin\gamma$

Given that $\cos\alpha + \cos\beta + \cos\gamma = 0$

$\sin\alpha + \sin\beta + \sin\gamma = 0$

Now $x+y+z = (\cos\alpha + i \sin\alpha) + (\cos\beta + i \sin\beta) + (\cos\gamma + i \sin\gamma)$

$= (\cos\alpha + \cos\beta + \cos\gamma) + i (\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i0$

$x+y+z = 0$ ....(1)

As we know that

$x_3 + y_3 + z_3 - 3xyz = (x+y+z)(x_2 + y_2 + z_2 - xy - yz - zx)$

$x_3 + y_3 + z_3 - 3xyz = 0$ (by (1))

$x_3 + y_3 + z_3 = 3xyz$

$(\cos\alpha + i \sin\alpha)_3 + (\cos\beta + i \sin\beta)_3 + (\cos\gamma + i \sin\gamma)_3 = 3 [(\cos\alpha + i \sin\alpha)(\cos\beta + i \sin\beta)$

$(\cos\gamma + i \sin\gamma)]$

$\cos 3\alpha + i \sin 3\alpha + \cos 3\beta + i \sin 3\beta + \cos 3\gamma + i \sin 3\gamma = 3[\cos (\alpha+\beta+\gamma) + i \sin(\alpha+\beta+\gamma)]$

$(\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma) = 3 \cos (\alpha+\beta+\gamma) + i\, 3 \sin(\alpha+\beta+\gamma)$

Comparing both sides, we get

$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos (\alpha+\beta+\gamma)$

Chapter - 2
Matrices

.1 **Basic Concept :**

Definition

A system of 'mn' numbers arranged in m rows and n columns and bounded by the brackets [ ] is called as m by n matrix, written by mxn matrix. A matrix is also denoted by a single capital letter.

Example :

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ & & & & \\ & & & & \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{bmatrix}$$

Where a_{ij} is an element present in i^{th} raw and j^{th} column.

0.2 Type of Matrix

1. **Null matrix :** If all the entries / elements of a matrix are zero, it is called as a null or zero or void matrix and it is denoted by **'O'.**

Example : $$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4}$$

2. **Row matrix :** A matrix having a single row is called a row matrix.

Example : $[1 \ \ 3 \ \ 5 \ \ 7]_{1\times 4}$

3. **Column matrix :** A matrix having a single column is called a column matrix.

Example : $$\begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix}_{4\times 1}$$

4. **Square matrix :** If the number of rows is equal number of columns, matrix is known as square matrix.

Example : $$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}_{3\times 3}$$

Note : A matrix is not a square matrix is called as rectangular matrix.

5. **Diagonal matrix :** A matrix having non diagonal elements are zero is called diagonal matrix.

Example : $\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & & a_{33} & & \\ 0 & & & a_{44} & \\ 0 & 0 & 0 & & a_{nn} \end{bmatrix}_{n\times n}$

6. Scalar matrix : if the diagonal elements of the diagonal matrix are same then the matrix is called as scalar matrix.

Example : $[a] = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

7. Unit matrix / scalar matrix :

A scalar matrix is said to be a unit matrix if all the elements in the leading diagonal are unity.

Example : $I_1 = [1]$ $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. Triangular matrix :

It is of two types

i) Lower Triangular matrix

ii) Upper Triangular matrix

Lower Triangular matrix : A square matrix (a_{ij}) is called a lower triangular matrix if a_{ij} = 0, i<j i..e elements above the leading diagonal are zero.

Example : $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 7 & 2 & 4 \end{bmatrix}$

Upper Triangular matrix : A square matrix (a_{ij}) is called an upper triangular matrix if a_{ij} = 0, i>j i.e. elements below the leading diagonal are zeros.

Example : $\begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

0.3 Procedure to find Minor of a Matrix :

If we select any 'r' rows and 'r' columns from a matrix A by deleting all other rows and columns, then the determinant formed by these r x r elements is called minor of 'A' of order 'r'.

A matrix 'A' having minors of same order

Example : $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}_{4\times 4}$

The different minors of order 3 of matrix 'A' are

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 3 \\ 3 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 & 3 \\ 3 & 2 & 1 \\ 6 & 8 & 7 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 & 1 \\ 6 & 8 & 7 \\ 1 & 2 & 3 \end{bmatrix}$$

(deleting 4^{th} row a 4^{th} column)

$$\begin{bmatrix} 3 & 2 & 1 \\ 6 & 8 & 7 \\ 2 & 4 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & 0 \\ 4 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 4 & 3 & 2 \\ 2 & 1 & 3 \\ 8 & 7 & 5 \end{bmatrix}$$

And so on

1.1 **Rank of Matrix**

Definition :

A matrix is said to be of rank 'r' when it has at least one non-zero minor of order 'r' and every minor of order higher than 'r' vanishes.

or

The rank of a matrix is the largest order of any non-vanishing minor of the matrix.

Rank of matrix 'A' is denoted as $\rho(A)$

Note :

1. Rank of a null matrix is equal to zero
2. Rank of a identity matrix is same as it's order.
3. Rank of a unit matrix is same as its order.
4. Rank of matrix A and Rank of matrix A^T are equal i.e. $\rho(A) = \rho(A^T)$

1.2 Elementary transformation of a matrix

By an elementary transformation of matrix, the following operations are hold good.

1. Interchange of any two rows or columns.
2. The multiplication of any row or column by a nonzero number.
3. The addition of a constant multiple of the elements of any row or column to the corresponding elements of any other row or column.

Notation :

(i) R_{ij} for interchanging of i^{th} row (R_i) and j^{th} rows (R_j)

(ii) KR_i for multiplication of i^{th} row by k.

(iii) $R_i + KR_j$ for addition of i^{th} row with k times the j^{th} row.

Note : Elementary transformations do not change either the order or rank of a matrix.

1.2.1 Working procedure for Evaluating Rank of a matrix

Step-1 : Convert the given matrix to triangular matrix using elementary transformations.

or

Use row reduce echelon form of matrix.

Step-2 : The no. of non zero rows is the rank of the matrix.

Example : Find the Rank of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

Ans. : Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ $\quad R_2 \to R_2 - R_1 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 2 & 6 & 5 \end{bmatrix}$

$R_3 \to R_3 - 2R_1 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$ $\quad R_3 \to R_3 - R_2 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Hence, $\rho(A) = 2$

A Consistency of Linear system of equation :

The system of 'm' linear equations with 'n' unknown x_1, x_2, x_3 - - - x_n are in the form of

$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + - - - + a_{1n} x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + - - - + a_{2n} x_n = b_2$

- - - - - - - - - - - - - - - - - - -

- - - - - - - - - - - - - - - - - - -

$a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + - - - + a_{mn} x_n = b_n$

Where co-efficient matrix A = $\begin{bmatrix} a_{11} & a_{12} & - & - & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & - & - & a_{2n} \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ a_{m1} & a_{m2} & - & - & - & - & a_{mn} \end{bmatrix}$

Where Augmented matrix K= $\left[\begin{array}{cccccc|c} a_{11} & a_{12} & a_{13} & . & . & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & . & . & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & . & . & a_{3n} & b_3 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & . & . & a_{mn} & b_n \end{array}\right]$

If the system of linear equation has a solutions. It is called as consistent system, otherwise the system is called in consistent.

1.3 Rouches Theorem : The system of linear equations is consistent if and only if the co-efficient matrix 'A' and the augmented matrix K are of the same rank otherwise the system is inconsistent.

1.4 Working Procedure : To Test the consistency and find the solution.

Step-1 : Express the given system of equation in matrix form.

Step-2 : Write the co-efficient matrix (A) and augmented matrix (K)

Step-3 : Convert the matrix 'A' and 'K' to a triangular matrix using elementary transformation method.

Step-4 : Determine $\rho(A)$ and ρ (k)

Step-5 :

Case -1 : If $\rho(A) = \rho(k)$ = no. of variable then given system of equation is consistent i.e. having unique solution.

Else if $\rho(A) = \rho(K)$ < no. of variable the system having infinitely many solution.

Case-2 : if $\rho(A) \neq \rho(K)$, the given system of equation is inconsistent i.e. have no solution.

Step-6 : If the system is consistent convert the reduce matrix form to equation form and solve the equation to find the solutions.

Example : 1 Test for consistency and solve.

$5x + 3y + 7z = 4$

$3x + 26y + 2z = 9$

$7x + 2y + 10z = 5$

Solution :

Step-1 : The given system of linear equation is of the form

$$\begin{pmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}$$

Step-2 : Where coefficient Matrix, A = $\begin{pmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{pmatrix}$

And Augmented Matrix, K = $\left(\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right)$

Step-3 : K = $\left(\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right)$

$$R_2 \to \underset{\sim}{5R_2 - 3R_1} \left(\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 7 & 2 & 10 & 5 \end{array}\right) \qquad \begin{array}{rcrrr|r} 3R_1 & = & 15 & 9 & 21 & 12 \\ 5RL2 & = & 15 & 130 & 10 & 45 \end{array}$$

$$R_2 \underset{\sim}{\to} \frac{R_2}{11} \left(\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 7 & 2 & 10 & 5 \end{array}\right)$$

(To make 1st entry of 2nd row to zero)

$$R_3 \underset{\sim}{\to} 7R_1 - 5R_3 \left(\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 7 & 11 & -1 & 3 \end{array}\right) \qquad \left\{\begin{array}{rcrrr|r} 7R_1 & = & 35 & 21 & 49 & 28 \\ 5R_3 & = & 35 & 10 & 50 & 25 \end{array}\right.$$

(To make 1st entry of 3rd row to zero)

$$R_3 \underset{\sim}{\to} R_3 - R_2 \left(\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Step-4 : So ρ(A) = 2 and ρ(K) = 2

Step-5 : Since ρ (A) = ρ (K) = 2 < no. unknowns (3). By Rouches theorem, the system is consistent and have many solution.

Step-6 : Hence the reduced matrix is $\begin{pmatrix} 5 & 3 & 7 \\ 3 & 11 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$

Again converting to equation form (start with lowest row)

11y – z = 3

$\Rightarrow 11y = z + 3$

$\Rightarrow y = \frac{z}{11} + \frac{3}{11}$ ………….. (i) y in terms of z.

and 5x + 3y +7z = 4

$\Rightarrow 5x = -3y - 7z + 4$

$\Rightarrow x = -\frac{3}{5}y - \frac{7}{5}z + \frac{4}{5}$

Putting the value of y,

$$x = -\frac{3}{5}\left(\frac{z}{11} + \frac{3}{11}\right) - \frac{7}{5}z + \frac{4}{5}$$

$$= -\frac{3z}{55} - \frac{9}{55} - \frac{7}{5}z + \frac{4}{5}$$

$$= \frac{-3z - 77z}{55} - \left(\frac{9 - 44}{55}\right)$$

$$= \frac{-80z}{55} + \frac{35}{55}$$

$$= \frac{-16z}{11} + \frac{7}{11}$$

The solution is $\frac{7}{11} - \frac{16}{11}z, \frac{z}{11} + \frac{3}{11}, z$

For each value of z, there is a solution of the given system. In particular putting z = 0

$\left(\frac{7}{11}, \frac{3}{11}, 0\right)$ is a particular solution. Similarly by putting different value of z, we will get different solution.

Example : 2

Investigate the values of λ & μ so that equations

2x + 3y + 5z = 9

$7x + 3y - 2z = 8$

$2x + 3y + \lambda z = \mu$

Have (i) no solution (ii) a unique solution (iii) an infinite no. of solution.

Solution : Expressing the given system of linear equation to matrix form

$$\begin{pmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \\ \mu \end{pmatrix}$$

Here $A = \begin{pmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{pmatrix}$ and $k = \left(\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right)$

Now $\left(\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right) \underset{\sim}{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right) \sim$

Case-1

If $\lambda - 5 = 0$ and $\mu - 9 \neq 0$

i.e $\lambda = 5$ and $\mu \neq 9$

Thus $\rho(A) \neq \rho(K)$. So the system have no solution.

Case-2

If $\lambda - 5 \neq 0$ i.e $\lambda \neq 5$, for all values of μ.

$\rho(A) = \rho(K) = 3$ (no. of unknowns)

The system have unique solution.

Case-3

If $\lambda - 5 = 0$ and $\mu - 9 = 0$

i.e $\lambda = 5$ and $\mu = 9$

$\rho(A) = \rho(K) = 2 <$ no. of unknowns.

The system have infinite no of solutions.

Important Question with Solution

1. Fin the rank of the matrix

$$\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

Ans. : Step-1 : Given matrix $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

Using elementary transformation, the matrix A becomes

$$\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ -3 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 + \frac{R_2}{2}} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is a upper triangular matrix

Step-2 : Hence ρ(A)=2 (no. of non zero rows are 2).

2. Investigate for what values of λ and μ the simultaneous equation

X + y + z = 6

x + 2y + 3z =10

x + 2y + λz = μ

Have (a) no solution

(b) unique solution

(c) an infinite solution.

Solution :

Step-1 : Express given equation to matrix form .

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Where A= $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}$, $K = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$

K = $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 1 & 2 & \lambda & \mu \end{array}\right]$

$$\xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 3 & \mu - 10 \end{array}\right]$$

Which is a upper triangular matrix.

Step-2 : The matrix K has different cases

Case-I : If $\lambda - 3 = 0$ and $\mu - 10 \neq 0$

$\Rightarrow \lambda = 3$ and $\mu \neq 10$

Thus ρ(A) = 2 and ρ(K) = 3

i.e. ρ(A) $\neq$ ρ(K)

So the system has no solution.

Case-II : If $\lambda - 3 \neq 0$ and $\mu - 10 \neq 0$

$\Rightarrow \lambda \neq 3$ and $\mu \neq 10$

Thus ρ(A) = 3 = ρ(K)

So the system has unique solution.

Case-III : If λ – 3 = 0 and μ – 10 = 0

$\Rightarrow \lambda = 3$ and $\mu = 10$ Thus $\rho(A) = 2$ and $\rho(K) = 2$

i.e $\rho(A) = \rho(K) = 2 <$ no. of variable

So the system has infinite no. of solution.

3. Test the consistency and solve

2x – 3y + 7z = 5

3x + y – 3z = 13

2x + 194 – 47z = 32

Ans. : Step-1 : The given system is in the form of Ax = B

$$\Rightarrow \begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

Step-2 : Where $A = \begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}$, $K = \left[\begin{array}{ccc|c} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right]$

Step-3 : Using elementary row transformation

$$K = \left[\begin{array}{ccc|c} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right] \xrightarrow{R_3 \to 2R_2 - 3R_1} \left[\begin{array}{ccc|c} 2 & -3 & 7 & 5 \\ 0 & 11 & -27 & 11 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_2 - R_1} \left[\begin{array}{ccc|c} 2 & -3 & 7 & 5 \\ 0 & 11 & -27 & 11 \\ 0 & 22 & -54 & 27 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 2 & -3 & 7 & 5 \\ 0 & 11 & -27 & 11 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

Step-4 : So, $\rho(A) = 2$ and $\rho(K) = 3$

Step-5 : Since $\rho(A) \neq \rho(K)$

So, by Rouche's theorem, the system is in consistent, having no solution.

4. Test the consistency and solve

5x + 3y + 7z = 4

3x + 26y + 2z =9

7x + 2y + 10z = 5

Solution :

Step – I : The given system of linear equation is in the form AX = B

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Step-2 : Where the co-efficient matrix

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$$

Augmented matrix K = $\left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$

Step-3 : To convert 'K' in to upper triangular matrix using elementary transformation.

$$K=\left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right] \xrightarrow{R_3 \to 5R_2 - 3R_1} \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

$$\xrightarrow{R_3 \to 5R_3 - 7R_1} \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{array}\right] \xrightarrow{R_3 \to 11R_3 + R_2} \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Step-4 : So, ρ(A) = 2 and ρ(K) = 2

Step-5 : Since ρ(A) = ρ(K) = 2 < no. of variable

So by Rouches theorem, the system is consistent and having infinitely many solution.

Step-6 : $\begin{bmatrix} 5 & 3 & 7 \\ 0 & 121 & -11 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 33 \\ 0 \end{bmatrix}$

By using matrix multiplication method. We have

5x + 3y + 7z = 4, 121y – 11z = 33,

$\Rightarrow$ 11(11y – z) = 33

$\Rightarrow$ 11y – z = 3

$\Rightarrow$ 11y = 3 + z

$\Rightarrow$ $y = \frac{3+z}{11}$

$y = \frac{3}{11} + \frac{z}{11}$ --- (1)

Putting equation (1) in 5x + 3y + 7z = 4

$\Rightarrow$ $5x + 3\left(\frac{3}{11} + \frac{z}{11}\right) + 7z = 4$

$\Rightarrow$ $5x + \frac{9}{11} + \frac{3z}{11} + 7z = 4$

$\Rightarrow$ $5x + \frac{9 + 3z + 77z}{11} = 4$

$\Rightarrow$ $5x = 4 - \frac{80z + 9}{11}$

$\Rightarrow$ $5x = \frac{44 - 80z - 9}{11}$

$\Rightarrow$ $= \frac{35 - 80z}{55}$

$\Rightarrow$ $\frac{35}{55} - \frac{80z}{55}$

$\Rightarrow \dfrac{7}{11}-\dfrac{16}{11}z$

So, solution is $\dfrac{7}{11}-\dfrac{16z}{11},\ \dfrac{3}{11}+\dfrac{z}{11},\ z$ for each value of z, there is a solution, of the given equation.

In particular z = 0 $\left(\dfrac{7}{11},\dfrac{3}{11},0\right)$ is a particular solution.

Similarly for different value's of z, we get different solution.

5. Show that if $\lambda \neq -5$, the system of equation

$3x - y + 4z = 3$

$x + 2y - 3z = -2$

$6x + 5y + \lambda z = -3$

Have a unique solution, if $\lambda = -5$ show that the equation are consistent and determine the solution is each cash.

Ans.: The given system of linear equation is of the form AX = B

Step -1 :

$$\Rightarrow \begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$$

Step -2 : Where the co-efficient matrix

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix}$$

and Augmented matrix $K = \left[\begin{array}{ccc|c} 3 & -1 & 4 & 3 \\ 1 & 2 & -3 & -2 \\ 6 & 5 & \lambda & -3 \end{array}\right]$

Step-3 : (To convert K matrix into upper triangular matrix using elementary transformation)

$$K = \left[\begin{array}{ccc|c} 3 & -1 & 4 & 3 \\ 1 & 2 & -3 & -2 \\ 6 & 5 & \lambda & -3 \end{array}\right] \xrightarrow{R_2 \to 3R_2 - R_1} \left[\begin{array}{ccc|c} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 6 & 5 & \lambda & -3 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|c} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 7 & \lambda-8 & -9 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 0 & \lambda+5 & 0 \end{array}\right]$$

Step-4 : It has two cases

Case-I : If $\lambda + 5 = 0 \Rightarrow \lambda = -5$

So, $\rho(A) = 2$ and $\rho(K) = 2$

So, $\rho(A) = \rho(K) = 2 <$ no. of variable

So by Rouches theorem, the system inconsistent and having infinite solution.

Case-II : If $\lambda + 5 \neq 0 \Rightarrow \lambda \neq -5$

So $\rho(A) = 3$ and $\rho(K) = 3$

Since $\rho(A) = \rho(K) =$ no of variable

So by Ruches theorem, the system is consistent and have a unique solution.

Step-5 : To find solution for case-I

$$\begin{bmatrix}3 & -1 & 4\\0 & 7 & -13\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\-9\\0\end{bmatrix}$$

By using matrix multiplication method, we get

$3x - y + 4z = 3,$ $\qquad$ $7y - 13z = -9$

$3x - y + 4z = 3$ $\qquad$ $\Rightarrow 7y = -9 + 13z$

$\Rightarrow 3x - \left(\frac{-9}{7} + \frac{13}{7}z\right) = 3 - 4z$ $\qquad$ $\Rightarrow y = \frac{-9}{7} + \frac{13}{7}z$

$\Rightarrow 3x + \frac{9}{7} - \frac{13z}{7} = 3 - 4z$

$\Rightarrow 3x = 3 - \frac{9}{7} + \frac{13z}{7} - 4z$

$\Rightarrow 3x = \frac{21 - 9 + 13z - 28z}{7}$

$\Rightarrow 3x = \frac{12 - 15z}{7}$ $\qquad$ $\Rightarrow x = \frac{4}{7} - \frac{5}{7}z$

So the solution is $\frac{4}{7} - \frac{5z}{7}, \frac{-9}{7} + \frac{13}{7}z, z$

To find solution for case-2

$$\begin{bmatrix}3 & -1 & 4\\0 & 7 & -13\\0 & 0 & \lambda+5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\-9\\0\end{bmatrix}$$

By using matrix multiplication method, we have

$3x - y + 4z = 3,$ $\qquad$ $7y - 13z = -9,$ $\qquad$ $(\lambda + 5)z = 0$

$\Rightarrow 7y - 13z = -9$ $\qquad$ $\Rightarrow z = 0$

$\Rightarrow 7y - 13o = -9$

$\Rightarrow y = \frac{-9}{7}$

$3x - y + 4z = 3$

$\Rightarrow 3x - \left(\frac{-9}{7}\right) + 4 = 3$

$\Rightarrow 3x + \frac{9}{7} = 3$

$\Rightarrow 3x = 3 - \frac{9}{7} = \frac{21 - 9}{7} = \frac{12}{7}$

$\Rightarrow x = \frac{4}{7}$

6. For what values of K, the equation

$X + y + z = 1$

$2x + y + 4z = K$

$4x + y + 10z = K^2$

Have a solution and solve them completely in each cases.

Solution :

Step-1 : The given system of linear equation is of the form AX = B

$$\begin{bmatrix}1 & 1 & 1\\2 & 1 & 4\\4 & 1 & 10\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}I\\K\\K^2\end{bmatrix}$$

Step-2 : Co efficient matrix A = $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix}$

And Augmented matrix K = $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & K \\ 4 & 1 & 10 & K^2 \end{array}\right]$

Step-3 : K = $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & K \\ 4 & 1 & 10 & K^2 \end{array}\right] \xrightarrow{R_2 \to R_1 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & K-2 \\ 4 & 1 & 10 & K^2 \end{array}\right]$

$\xrightarrow{R_3 \to R_3 - 4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & K-2 \\ 0 & -3 & 6 & K^2-4 \end{array}\right] \xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & K-2 \\ 0 & 0 & 0 & K^2-3K+2 \end{array}\right]$

Step-4 : Since the given system has a solution.

$\rho(A) = \rho(K)$

$\Rightarrow K^2 - 3K + 2 = 0$

$\Rightarrow (K-2)(K-1) = 0$

$\Rightarrow K = 2,1$

For K = 2

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & K-2 \\ 0 & 0 & 0 & K^2-3K+2 \end{array}\right]$

$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\therefore$ Solution of the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

By using matrix multiplication method we have

$x + y + z = 1,$ $\quad\quad$ $-y + 2z = 0,$

$\Rightarrow x + 2z + z = 1$ $\quad\quad$ $\Rightarrow 2z = y$

$\Rightarrow x + 3z = 1$

$\Rightarrow x = 1 - 3z$

So solution is (1 – 3z, 2z, z)

For K = 1

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$x + y + z = 1,$ $\quad\quad$ $-y + 2z = -1,$

$\Rightarrow y = 2z + 1$

$\Rightarrow x + 2z + 1 + z = 1$

$\Rightarrow x + 3z = 0$

$\Rightarrow x = -3z$

So solution is (- 3z, 1 + 2z, z)

In particular z = 0 solution is (0, 1, 0)

Chapter-3

Linear differential equation

3.1 Linear differential equation with constant coefficient

Definition : Linear differential equation is an equation in which the dependent variable and its derivatives occur only in first degree and are not multiplied together.

General equation of order 'n' is of the form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + ... + p_n y = X \quad .... (i)$$

Where p_1, p_2 …. p_n are all constant coefficients and 'X' is a function of x or zero. Such differential equations are most important in the study of electro mechanical vibration and other Engineering problems.

Operator 'D'

'D' denotes the differential operator

$$\text{i.e. } D = \frac{d}{dx},\ D^2 = \frac{d^2}{dx^2},\ D^3 = \frac{d^3}{dx^3}\ ...$$

$$\text{i.e } Dy = \frac{dy}{dx},\ D^2 y = \frac{d^2 y}{dx^2},\ D^3 y = \frac{d^3 y}{dx^3} \text{ and ... so on.}$$

Using these notations, equation (i) becomes

$D^n y + P, D^{n-1}y + P_2 D^{n-2}y + ... + P_n y = X$

$\Rightarrow$ $(D^n + P_1 D^{n-1} + P_2 D^{n-2} + ... + P_n)\, y = X$

$\Rightarrow$ $F(D)y = X$ .... (1)

Where $f(D) = D^n + P_1 D^{n-1} + ... + P_n$ which is a polynomial in D

Thus the symbol 'D' stands for the operation of differentiation and can be treated as an algebraic quantity i.e. f(D) can be factorised by ordinary rules of algebra and the factors may be taken in order such as $D^3 + D^2 + D$ + constant.

Types of linear differential equation (L.D.E)

The linear differential equation (i)

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + P_n y = X$$

are of two types i.e. (A) Homogeneous L. D. E.

(B) Non-homogeneous L. D. E.

(A) Homogeneous linear differential equation

The linear differential equation (i) is said to be homogeneous linear differential equation if X = 0

So equation (i) becomes

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + ... P_n y = 0$$

(B) Non homogeneous linear differential equation :-

The linear D.E. (i) is said to be non-homogeneous, if X $\neq$ 0 i.e.

i.e. $\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+.....+P_ny=X,\ \ x\neq 0$

3.2 <u>General solution of linear equation in terms of C.F. & P.I</u>

The general solution of L.D.E is

C.S = C. F + P. I where C.F$\rightarrow$ complimentary function.

P. I. $\rightarrow$ particular integral

C. S $\rightarrow$ complete solution

3.3 : Solution of homogeneous linear differential equation

The solution of homogeneous linear differential equation

$$\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+.....+P_ny=0, \quad .... \text{(ii)}$$

Is called as complimentary function (C.F.) and it is denoted as 'y_c'

In this case C.S. = C.F as there is no P.I.

To solve the equation (ii)

Applying Operator 'D', $\left(D=\frac{d}{dx}\right)$

$\Rightarrow$ $(D^n + P_1D^{n-1} + ...+P_n)y = 0$

$\Rightarrow$ $(D^n + P_1D^{n-1} + ...+P_n) = 0$ ……………………….. (iii)

Which is called <u>auxiliary equation</u> (A.E) or <u>characteristic equation</u>.

Now solving equation (iii) we obtain different values of D which are known as roots of the auxiliary equation Let D = m_1, m_2, m_3 …. m_n be roots of A.E.

Then solution is y = $C_1e^{m_1x}+C_2e^{m_2x}+C_3e^{m_3x}+....+C_ne^{m_nx}$

Working procedure to find solution of homogeneous differential equation.

Step-1 : Convert the given equation into 'D' operator form.

Step-2 : Make the equation in the form of f(D) y = 0

Step-3 : Find the roots of the auxiliary equation f(D) = 0

Step-4 : Take the value of D as m_1, m_2 … and so on.

There are 4 cases

Case-I : (Roots are real and distinct)

Let the roots are $m_1 \neq m_2 \neq m_3$ …..

$$y=C_1e^{m_1x}+C_2e^{m_2x}+C_3e^{m_3x}+....$$

Case-II : (Roots are real and equal)

a) If the two roots are equal i.e $m_1 = m_2$, m_3 …..

$$y_c=(c_1+c_2x)e^{m_1x}+c_3e^{m_3x}+...$$

b) If three roots are equal $m_1 = m_2 = m_3, m_4 \ldots$

$$y_c = (c_1 + c_2 x + c_3 x^2)e^{m_1 x} + c_4 e^{m_4 x}$$

Case-III : (If roots are complex) like $(m_1 = \alpha \pm \beta i), m_2 \ldots$

$$y_c = e^{\alpha x}[A\cos\beta x + B\sin\beta x] + c_1 e^{m_2 x} + \ldots$$

Case-IV : (Roots are complex and repeated) like $m_1 = m_2 = \alpha \pm i\beta$, m_3

Then $y_c = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] + C_5 e^{m_3 x}$

Example : Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Solution : given D.E is $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Step-1 : Above equation in symbolic form $D^2y - 5Dy + 6y = 0$

Step-2 : $(D^2 - 5D + 6)y = 0$

Step-3 : The A.E equation is

$D^2 - 5D + 6 = 0$

$\Rightarrow D^2 - 3D - 2D + 6 = 0 \quad \Rightarrow D(D - 3) - 2(D - 3) = 0$

$\Rightarrow (D - 3)(D - 2) = 0 \quad \Rightarrow D = 3, 2$

Step-4 : Here $m_1 = 3$, $m_2 = 2$ (Roots are real and distinct)

$\therefore y_c = c_1 e^{3x} + c_2 e^{2x}$

Where c_1 and c_2 are arbitrary constants.

Example - 2

Solve $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$

Solution : The given D.E. is $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$

Step-1 : Above equation can be written in symbolic form,

$D^2y + 6Dy + 9y = 0$

Step-2 : $(D^2 + 6D + 9)y = 0$

Step-3 : The A.E. is $D^2 + 6D + 9 = 0$

$\Rightarrow (D + 3)^2 = 0$

$\Rightarrow D = -3, -3$

Step-4 : Here $m_1 = m_2 = -3$ (Real and repeated roots)

$\therefore y_c = (c_1 + c_2 t)\, e^{-3t}$

Where c_1 and c_2 are arbitrary constants.

Example-3

Solve $\frac{d^2y}{dx^2} + y = 0$

Solution : The given D.E. is $\frac{d^2y}{dx^2} + y = 0$

Step-1 : Above equation can be written in symbolic form, $D^2y + y = 0$

Step-2 : $\Rightarrow (D^2 + 1)y = 0$

Step-3 : The A. E is $D^2 + 1 = 0$

$\Rightarrow D^2 = -1$

$\Rightarrow D = \pm i = 0 \pm i$

Step-4 : Here m = $0 \pm i$ (Imaginary roots) (i.e.α=0 & β=1)

$\therefore y_c = e^{0x} (A \cos x + B \sin x)$

$\Rightarrow y_c = A \cos x + B \sin x$

Where A & B are arbitrary constants.

Example – 4

Solve $(D^2 - 2D + 4)^2 y = 0$

Solution :

Step -1 : The given D.E. is $(D^2 - 2D + 4)^2 y = 0$

Step-2 : The A. E is

$(D^2 - 2D + 4)^2 = 0$

$\Rightarrow D^2 - 2D + 4 = 0$

$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{2 \pm \sqrt{4 - 4 \times 4}}{2} = \frac{2 \pm \sqrt{-12}}{2}$

$= \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm \sqrt{3}i, \quad 1 \pm \sqrt{3}i$

Step -3 :

$\therefore m_1 = m_2 = 1 \pm \sqrt{3}i$

Here α=1, β=$\sqrt{3}$

$\therefore \quad y_c = e^x \left[(c_1 + c_2 x)\cos\sqrt{3}x + (c_3 + c_4 x)\sin\sqrt{3}x\right]$

Where c_1, c_2, c_3, c_4 are arbitrary constant.

Solution of Non-hamogeneous linear dliff. Equation

The solution of non-homogeneous diff. equation is known as complete solution which is written as

y = C. F + P.I

or $y = y_c + y_p$

Note : The C. F. (Y_C) may be obtained as explained in 2.2. (A).

3.3 Rules of find Particular Integral (P.I)

Definition : The non-homogeneous linear differential equation is

$$\left(\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_n y\right) = X$$

$\Rightarrow$ $(D_n + P,\ D^{n-1} + \ldots + P_n)y = X$

$\Rightarrow$ $f(D)y = X$

$\Rightarrow$ $y = \frac{1}{f(D)}X$, [which resulting the function of x not containing arbitrary constant which when operated upon by f(D) gives X, i.e. $f(D)\left\{\frac{1}{f(D)}X\right\} = X$]

$\Rightarrow$

i.e P.I = $\frac{1}{f(D)}X$

There are different cases to find P.I depending upon 'X'

Case-1 : When X = e^{ax}

y_p = P.I = $\frac{1}{f(D)}e^{ax}$, put D = a in f(D)

then y_p = $\frac{1}{f(a)}e^{ax}$, Provided $f(a)\neq 0$

If f(a) = 0, Rule fails

Then find f' (D) (1st derivative of f(D)) and put D = a in f' (D)

Then y_p = X $\frac{1}{f'(a)}e^{ax}$, Provided $f'(a) \neq 0$

Again if f' (a) = 0, Rule fails, then find f'' (D) (Second derivative of f(D))

and y_p= $x^2 \frac{1}{f''(a)}e^{ax}$, provided $f''(a) \neq 0$ and so on.

Case-II :

When X = Sin (ax + b) or Cos (ax+b)

y_p = P.I = $\frac{1}{f(D^2)}\text{Sin}(ax+b)$ or cos(ax+b)

[Here $f(D^2)$ means it may contains the term D^2 and higher order than D^2 and lower order of D]

Put $D^2 = -a^2$ in $f(D^2)$

If $f(-a^2) \neq 0$

Then y_p = $\frac{1}{f(-a^2)}\sin(ax+b)$

If $f(-a^2) = 0$, Rule fails, then find $f'(D^2)$ (1^{st} derivative of f(D)) and put $D^2 = -a^2$ in $f'(D^2)$

…….If $f'(-a^2) \neq 0$ then

$$y_p = x\frac{1}{f'(-a^2)}\sin(ax+b)$$

Again if $f'(-a^2) = 0$, Rule fails

Their $y_p = x^2\frac{1}{f''(-a^2)}\sin(ax+b)$; provided $f''(-a^2) \neq 0$ and so on

Case-III : When X = x^m (i.e.x, x^2, x^3 ……………)

$$y_p = P.I = \frac{1}{f(D)}x^m = [f(D)]^{-1}x^m$$

Convert f(D) into $\{1+\phi(D)\}$ or $\{1-\phi(D)\}$ (If possible) and then by using Binomial theorem we find solution.

Note : Applying Binomial expansion on $[f(D)]^{-1}$

$(1+D)^{-1} = (1-D+D^2-D^3+D^4 \ldots)$

$(1-D)^{-1} = (1+D+D^2+D^3 \ldots)$

To find y_p refer Q.4 (Imp. Question)

Case-IV : When X = e^{ax}.V, V being a function of X.

$$y_p = \frac{1}{f(D)}x = \frac{1}{f(D)}e^{ax}.V = e^{ax}.\frac{1}{f(D+a)}V$$

to find y_p operate $\frac{1}{f(D+a)}$ on V by using previous rule.

Working Procedure for solving linear differential equation

Step-1 : Write the given homogeneous differential equation.

Write A.E

Step-2 : Solve it for D and find the roots.

Step-3 : Find C.F. or y_c.

Step-4 : Find P.I or y_p.

Step-5 : Complete solution y = C. F + P.I

or $y = y_c + y_p$

Example-5

Solve $(D^2 + 5D + 6)\, y = e^x$

Solution : Step-1

Given differential equation is

$(D^2 + 5D + 6)\, y = e^x$

To find C.F

The homogeneous part of given differential equation is

$(D^2 + 5D + 6)\, y = 0$

$\Rightarrow D^2 + 5D + 6 = 0$

Which is a auxiliary equation.

Step-2

$D^2 + 5D + 6 = 0$

$\Rightarrow D^2 + 3D + 2D + 6 = 0$

$\Rightarrow D(D+3) + 2(D+3) = 0$

$\Rightarrow (D+3) + (D+2) = 0$

$\Rightarrow D+3 = 0,\; D+2 = 0$

$\Rightarrow D = -3, -2$

Step-3

C. F $= y_c = C_1e^{-3x} + C_2e^{-2x}$

Step-4

To find particular integral (P.I)

P.I $= y_p = \dfrac{1}{(D^2 + 5D + 6)} e^x$

Putting a = 1 in place D $= \dfrac{1}{1 + 5 \times 1 + 6} e^x = \dfrac{1}{12} e^x$

Step-5

Hence the complete solution is

$Y = y_c + y_p = C_1e^{-3x} + C_2e^{-2x} + \dfrac{1}{12} e^x$ (Ans.)

Example-6

Find P.I of $(D+2)(D-1)^2\, y = e^{-2x} + 2 \sin hx$

Solution : Given differential equation is

$(D+2)(D-1)^2\, y = e^{-2x} + 2 \sin hx$

$\Rightarrow$ PI $= y_p = \dfrac{1}{(D+2)(D-1)^2}\left(e^{-2x} + 2\sin hx\right)$

$$= \frac{1}{(D+2)(D-1)^2} \cdot \left(e^{-2x} + 2\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{(D+2)(D-1)^2} \cdot \left(e^{-2x} + e^x - e^{-x}\right)$$

$$\frac{1}{(D+2)(D-1)^2} \cdot e^{-2x} = \frac{1}{(D+2)} \cdot \left[\frac{1}{(D-1)^2} \cdot e^{-2x}\right]$$

$$= \frac{1}{D+2}\left[\frac{1}{(-2-1)^2} \cdot e^{-2x}\right]$$

$$= \frac{1}{D+2} \cdot \frac{1}{9} e^{-2x} \qquad \left(\frac{1}{dD}(D+2) = 1 \right)$$

$$= \frac{1}{9}.x.\frac{1}{1}.e^{-2x} = \frac{x}{9} e^{-2x}$$

$$\frac{1}{(D+2)(D-1)^2}.e^x = \frac{1}{(D-1)^2}\left[\frac{1}{(D+2)}.e^x\right]$$

$$= \frac{1}{(D-1)^2} \cdot \frac{1}{(1+2)}.e^x$$

$$= \frac{1}{3}.x^2 \frac{1}{2} e^x \qquad \left(\frac{d^2}{dD^2}(D-1)^2 = 2 \right)$$

$$= \frac{x^2 e^x}{6}$$

$$\frac{1}{(D+2)(D-1)^2}.e^{-x} = \frac{1}{(-1+2)(-1-1)^2}.e^{-x}$$

$$= \frac{e^{-x}}{4}$$

$$\therefore P.I = \frac{x}{9} e^{-2x} + \frac{x^2 e^{-x}}{6} + \frac{e^{-x}}{4}$$

Example-7 : Solve $y'' + 4y' + 4y = 3 \sin x + 4 \cos x$

Solution : Step-1 Given equation in symbolic form is $(D^2 + 4D + 4)y = 3 \sin x + 4 \cos x$

Step-2 : To find C.F

The homogeneous part of given differential equation is

$(D^2 + 4D + 4)y = 0$

Step-3 : $(D^2 + 4D + 4) = 0$

$\Rightarrow (D+2)^2 = 0$ (Repeated twice)

$\Rightarrow D = -2, -2$

Step – 4 : Here $m_1 = -2$, $m_2 = -2$

Step-5 : $y_c = (C_1 + C_2 x) e^{-2x}$

Step-6 : To find particular integral

$$y_p = \frac{1}{D^2 + 4D + 4}(3 \sin x + 4 \cos x)$$

Step-7 : $P.I = \frac{1}{D^2 + 4D + 4}(3 \sin x + 4 \cos x)$

$$= \frac{1}{-1^2 + 4D + 4}(3 \sin x + 4 \cos x)$$

$$= \frac{1}{4D+3}(3 \sin x + 4 \cos)$$

By multiplying (4D – 3) both in Numerator and Denominator.

$$= \frac{4D-3}{(4D+3)(4D-3)}(3\sin x + 4\cos x)$$

$$= \frac{4D-3}{16D^2-9}(3\sin x + 4\cos x)$$

$$= \frac{4D-3}{16(-1^2)-9}(3\sin x + 4\cos x)$$

$$= \frac{4D-3}{-25}(3\sin x + 4\cos x)$$

$$= \frac{-1}{25}\left[4D(3\sin x + 4\cos x) - 3(3\sin x + 4\cos x)\right]$$

D stands for differentiation.

$$= -\frac{1}{25}(12\cos x + 16(-\sin x) - 9\sin x - 12\cos x)$$

$$= -\frac{1}{25}(-25\sin x) = \sin x$$

Step-8 : So the complete solution is

$y = y_c + y_p$

$= (C_1 + C_2x)\, e^{-2x} + \sin x$

Example-8

Find P.I of $(D^2 + 6D + 3)y = e^{2x}$

Solution : P. I $= \frac{1}{D^2+6D+3}.e^{2x}$

$F(D) = D^2 + 6D + 3$

Put D=2, $F(2) = 2^2 + 6.2 + 3 = 19 \neq 0$

$$\therefore P.I = \frac{1}{2^2+6.2+3}e^{2x} = \frac{e^{2x}}{19}$$

Example-9

Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x$

Solution :

Step-1 : Given D.E is

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x$$

In symbolic form,

$D^3y - 3D^2y + 4Dy - 2y = e^x + \cos x$

$\Rightarrow (D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$

Step-2 : To find C.F.

The homogeneous part of given differential equation is

$\Rightarrow (D^3 - 3D^2 + 4D - 2)y = 0$

Step-3 : The A.E is

$D^3 - 3D^2 + 4D - 2 = 0$

$\Rightarrow D^3 - D^2 - 2D^2 + 2D + 2D - 2 = 0$

$\Rightarrow D^2 (D - 1) - 2 D (D - 1) + 2 (D - 1) = 0$

$\Rightarrow (D - 1) (D^2 - 2D + 2) = 0$

$\Rightarrow D - 1 = 0$ or $D^2 - 2D + 2 = 0$

$\Rightarrow D = 1,\ D = \dfrac{2 \pm \sqrt{-4}}{2} = \dfrac{2 \pm 2i}{2}$

$= 1 \pm i$

Step-4 : Here $m_1 = 1$, $m_2 = 1 \pm i$

$\therefore y_c = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x)$

Step – 5 : To find P. I.

$$y_p = \frac{1}{D^3 - 3D^2 + 4D - 2}(e^x + \cos x)$$

$$= \frac{1}{D^3 - 3D^2 + 4D - 2} e^x + \frac{1}{D^3 - 3D^2 + 4D - 2} \cos x$$

$$= x.\frac{1}{3D^2 - 6D + 4} e^x + \frac{1}{D^2.D - 3D^2 + 4D - 2} \cos x$$

$$= x.\frac{1}{3.1^2 - 6.1 + 4} e^x + \frac{1}{-1^2.D - 3(-1^2) + 4D - 2} \cos x$$

$$= \frac{xe^x}{1} + \frac{1}{-D + 3 + 4D - 2} \cos x$$

$$= xe^x + \frac{1}{3D + 1} \cos x$$

$$= xe^x + \frac{3D - 1}{(3D + 1)(3D - 1)} \cos x$$

$$= xe^x + \frac{3D - 1}{9D^2 - 1} \cos x$$

$$= xe^x + \frac{3D - 1}{9(-1^2) - 1} \cos x$$

$$= xe^x + \frac{3D - 1}{-9 - 1} \cos x$$

$$= xe^x + \frac{3D - 1}{-10} \cos x$$

$$= xe^x - \frac{1}{10}\{3D(\cos x) - 1.\cos x\}$$

$$= xe^x - \frac{1}{10}\{-3 \sin x - \cos x\}$$

$= xe^x + \frac{3}{10}\sin x + \frac{1}{10}\cos x$

Step-6

Hence the general solution is

$y = y_c + y_p$

$= c_1e^x + e^x(c_2\cos x + c_3\sin x) + xe^x + \frac{3}{10}\sin x + \frac{1}{10}\cos x$

Example-10

Find the P.I of $(D^3 + 1)y = e^x\cos x + \sin 3x$

Solution : P.I = $\frac{1}{D^3+1}\{e^x\cos x + \sin 3x\}$

$= \frac{1}{D^3+1}e^x\cos x + \frac{1}{D^3+1}\sin 3x$

$= e^x\frac{1}{(D+1)^3+1}\cos x + \frac{1}{(-3)^2.D+1}\sin 3x$

$= e^x\frac{1}{D^3+3D^2+3D+2}\cos x + \frac{1}{-9D+1}\sin 3x$

$= e^x\frac{1}{(-1^2).D+3(-1^2)+3D+2}\cos x + \frac{1+9D}{(1-9D)(1+9D)}\sin 3x$

$= e^x\frac{1}{-D-3+3D+2}\cos x + \frac{1+9D}{(1-81D^2}\sin 3x$

$= e^x\frac{1}{2D-1}\cos x + \frac{1+9D}{1-81(-3^2)}\sin 3x$

$= e^x\frac{2D+1}{(2D-1)(2D+1)}\cos x + \frac{1+9D}{1+729}\sin 3x$

$= e^x\frac{(2D+1)}{4D^2-1}\cos x + \frac{1}{730}(\sin 3x + 9D\sin 3x)$

$= e^x\frac{2D+1}{4(-1^2)-1}\cos x + \frac{1}{730}(\sin 3x + 27\cos 3x)$

$= \frac{e^x}{-5}(2D+1)\cos x + \frac{1}{730}(\sin 3x + 27\cos 3x)$

$= \frac{e^x}{-5}(2D\cos x + \cos x) + \frac{1}{730}(\sin 3x + 27\cos 3x)$

P.I = $\frac{e^x}{-5}(-2\sin x + \cos x) + \frac{1}{730}(\sin 3x + 27\cos 3x)$

Example-11

Solve $\frac{d^2y}{dx^2} + 9y = x\cos x$

Solution : Step-1

Given D. E is

$$\frac{d^2y}{dx^2}+9y=x\cos x$$

In symbolic form, $D^2y + 9y = x \cos x$

Step-2 $\Rightarrow (D^2 + 9)y = x \cos x$

Step-3 To find C.F.

The homogeneous part is $(D^2 + 9)y = 0$

A.E. is $(D^2 + 9) = 0$

$\Rightarrow D^2 = -9$

$\Rightarrow D = \pm 3i = 0 \pm 3i$

Step-4 Here $m = 0 \pm 3i$

$\therefore y_c = e^{ox} [c_1 \cos 3x + c_2 \sin 3x]$

$= c_1 \cos 3x + c_2 \sin 3x$

Step-5 To find P.I

$$y_p=\frac{1}{D^2+9}x\cos x$$

We know that $e^{ix} = \cos x + i \sin x$. Here Real part $\{e^{ix}\}$ = Re $\{e^{ix}\} = \cos x$ And Imaginary part $\{e^{ix}\}$ = Im $\{e^{ix}\} = \sin x$

$$=\text{Re}\left\{\frac{1}{D^2+9}xe^{ix}\right\}$$

$$=\text{Re}\left\{e^{ix}\frac{1}{(d+i)^2+9}x\right\}$$

$$=\text{Re}\left\{e^{ix}\frac{1}{D^2+2iD-1+9}x\right\} \qquad (\ i^2=-1)$$

$$=\text{Re}\left\{e^{ix}\frac{1}{D^2+2iD+8}x\right\}$$

$$=\text{Re}\left\{e^{ix}\frac{1}{8+D^2+2iD}x\right\}$$

$$=\text{Re}\left\{e^{ix}\frac{1}{8\left(1+\frac{D^2+2Di}{8}\right)}x\right\}$$

$$=\text{Re}\left\{\frac{e^{ix}}{8}\frac{1}{1+\frac{D^2+2Di}{8}}x\right\}$$

$$=\text{Re}\left\{\frac{e^{ix}}{8}\left(1+\frac{D^2+2Di}{8}\right)^{-1}x\right\}$$

$$= \text{Re}\left\{\frac{e^{ix}}{8}\left[1-\frac{D^2+2Di}{8}+\left(\frac{D^2+2Di}{8}\right)^2-\ldots\right]x\right\}$$

$$= \text{Re}\left\{\frac{e^{ix}}{8}\left(1-\frac{D^2+2Di}{8}\right)x\right\}$$

(Ignoring higher order derivatives for x)

$$= \text{Re}\left\{\frac{e^{ix}}{8}\left(x-\frac{2i}{8}Dx\right)\right\}$$

$$= \text{Re}\left\{\frac{e^{ix}}{8}\left(x-\frac{2i}{8}\right)\right\}$$

$$= \text{Re}\left\{\frac{xe^{ix}}{8}-\frac{ie^{ix}}{32}\right\}$$

$$= \text{Re}\left\{\frac{xe^{ix}}{8}\right\}-\text{Re}\left\{\frac{ie^{ix}}{32}\right\}$$

$$= \frac{x}{8}\text{Re}\{e^{ix}\}-\frac{1}{32}\text{Re}\{ie^{ix}\}$$

($ie^{ix} = i(\cos x + i\sin x) = i\cos x - \sin x = -\sin x + i\cos x$)

$$= \frac{x}{8}\cos x - \frac{1}{32}(-\sin x)$$

$$\Rightarrow y_p = \frac{x\cos x}{8}+\frac{\sin x}{32}$$

Hence the general solution is

y = C.F. + P.I

$$= C_1\cos 3x + C_2\sin 3x + \frac{x\cos x}{8}+\frac{\sin x}{32}$$

Partial Differential Equation

3.4 Define partial differential Equation

The differential equation in which more than one independent variable involves is called as partial differential equation.

Z = f(x, y), Here x, y are two independedent variable and z is dependent variable.

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial y^2} = t$$

$$\frac{\partial^2 z}{\partial x.\partial y} = \frac{\partial^2 z}{\partial y.\partial x} = s$$

Order : The 'order' of a partial differential equation is the order of highest partial derivatives in the equation.

Degree : Highest power of order of the partial differential equation is the 'degree' of partial differential equation.

Example :

i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}.\frac{\partial u}{\partial t}$

3.5 Formation of a. P. D. E

Let f(x, y, z, a, b) = 0 ……………….. (i)

From equation (i) eliminating both arbitrary constant and arbitrary function, we get partial differential equation.

Example : Form the partial differential equation by eliminating arbitrary constant.

$Z = ax + by + a^2 + b^2$

Ans. : $z = ax + by + a^2 + b^2$ ……………….. (i)

Differentiating partially to equation (i) with respect to x,

$$\frac{\partial z}{\partial x} = a \quad \Rightarrow p = a$$

Differentiating partially to equation (i) w.r. to y, we have

$$\frac{\partial z}{\partial y} = b \quad \Rightarrow q = b$$

$z = ax + by + a^2 + b^2$

$= px + qy + p^2 + q^2$

Example : Form the partial differential equation by eliminating arbitrary constant.

$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Ans : $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ ……………………… (i)

Differentiating partially both sides w.r.t x, to equation (i)

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2}$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{x}{a^2}$$

$$\Rightarrow \frac{1}{x}\frac{\partial z}{\partial x} = \frac{1}{a^2} \text{ ………………….. (2)}$$

Differentiating partially both sides w.r.t y to equation (i)

$$2\frac{\partial z}{\partial y} = \frac{2y}{b^2}$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{y}{b^2}$$

$$\Rightarrow \frac{1}{y}\frac{\partial z}{\partial y} = \frac{1}{b^2} \text{ ………………….. (3)}$$

Substituting the values of $\frac{1}{a^2}$ & $\frac{1}{b^2}$ from (2) and (3), in (1),

$$2z = x^2.\frac{1}{x}\frac{\partial z}{\partial x} + y^2\frac{1}{y}\frac{\partial z}{\partial y}$$

$\Rightarrow 2z = xp + yq$ which is the required equation.

Example :

Form differential equation $z = f(x^2 - y^2)$

Solution : Given $z = f(x^2 - y^2)$ ………………………… (1)

Differentiating partially w.r.t. x, equation (1),

$$\frac{\partial z}{\partial x} = \frac{\partial f(x^2 - y^2)}{\partial x}$$

$$\frac{\partial z}{\partial x} = 2xf'(x^2 - y^2)$$

$\Rightarrow$ $p=2x\ f'(x^2 - y^2)$ ………………… (2)

Differentiating partially w.r.t, y, equation (1),

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}f(x^2 - y^2)$$

$\Rightarrow q = -2yf'(x^2 - y^2)$ ………………… (3)

Dividing equation (2) by (3), we get

$$\frac{p}{q} = \frac{2xf'(x^2 - y^2)}{-2yf'(x^2 - y^2)}$$

$\Rightarrow \frac{p}{q} = -\frac{x}{y}$

$\Rightarrow yp = -xq$

$\Rightarrow$ yp + xq = 0 which is the required equation.

Example :

Form differential equation z = f(x+at) + g (x – at)

Solution : Given z = f (x + at) + g (x – at) ………………… (1)

Differentiating partially w.r.t. x, equation (1),

$$\frac{\partial z}{\partial x} = f'(x+at) + g'(x-at)$$

Again differentiating partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at) \quad \text{………………… (2)}$$

Differentiating partially w.r.t. t, equation (1),

$$\frac{\partial z}{\partial t} = af'(x+at) - a\, g'(x-at)$$

Again differentiating partially w.r.t t,

$$\frac{\partial^2 z}{\partial t^2} = a^2 f''(x+at) + a^2\ g''(x-at) \quad \text{………………… (3)}$$

$= a^2[f''(x+at) + g''(x-at)]$

From (2) & (3)

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

Which is the required solution.

3.6 Linear Equation of First Order :

A linear partial differential equation of the first order, commonly known as Lagrange's linear equation is of the form,

$Pp + Qq = R$ ……………… (1)

Where P, Q and R are function of x, y and z. When P, Q and R are independent of Z, it is known as linear equation.

Such a equation is obtained by an arbitrary function ϕ from $\phi(\mu, \nu) = 0$ Where u and ν are some functions of x, y, z.

Differentiate (2) partially w.r.t x and y.

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial \phi}{\partial \nu}\left(\frac{\partial \nu}{\partial x} + \frac{\partial \nu}{\partial z}p\right) = 0$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial \phi}{\partial \nu}\left(\frac{\partial \nu}{\partial y} + \frac{\partial \nu}{\partial z}q\right) = 0$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial \nu}$, we get

$$\begin{vmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p & \frac{\partial \nu}{\partial x}+\frac{\partial \nu}{\partial z}p \\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q & \frac{\partial \nu}{\partial y}+\frac{\partial \nu}{\partial y}q \end{vmatrix}=0$$

After simplification,

$$\left(\frac{\partial u}{\partial y}.\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}.\frac{\partial \nu}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}.\frac{\partial \nu}{\partial x}-\frac{\partial u}{\partial x}.\frac{\partial \nu}{\partial z}\right)q$$

$$=\frac{\partial u}{\partial x}\frac{\partial \nu}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial \nu}{\partial x} \quad \text{.................. (3)}$$

Now suppose u= a, ν = b where a, b are constants so that

$$\frac{\partial u}{\partial x}.dx+\frac{\partial u}{\partial y}.dy+\frac{\partial u}{\partial z}.dz=0$$

$$\Rightarrow du=0$$

$$\frac{\partial \nu}{\partial x}dx+\frac{\partial \nu}{\partial y}dy+\frac{\partial \nu}{\partial z}.dz=0$$

$$\Rightarrow dv=0$$

By cross multiplication, we have

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial \nu}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial \nu}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\frac{\partial \nu}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial \nu}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}.\frac{\partial \nu}{\partial y}-\frac{\partial u}{\partial y}.\frac{\partial \nu}{\partial x}}$$

$\Rightarrow \frac{dx}{p}=\frac{dy}{Q}=\frac{dz}{R}$ is the subsidiary equation.

The solution of these equation are u = a and ν = b

$\therefore \phi(u,\nu)=0$ is the required solution of equation (1)

Algorithim :

Step-1 : To form our problem into Pp + Qq = R.

Step-2 : Form the subsidiary equation

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

Step-3 : Choose multipliers P',Q',R' and P'',Q'',R''

in such a way $PP'+QQ'+RR'=0$

and $PP''+QQ''+RR''=0$

Step-4 :

$$\int P'dx+\int Q'dy+\int R'dz=C$$

$$\Rightarrow u(x,y,z)=c \quad \text{.................. (1)}$$

$$\int P''dx + \int Q''dy + \int R''dz = C$$

$$\Rightarrow \nu(x, y, z) = K \quad \text{.................. (2)}$$

Combining equation (1) and (2) we get desired result.

Example-1

Solve $x^2 (y - z)p + y^2 (z - x)q = z^2(x - y)$

Solution :

Step-1 : Given D. E is

$X^2(y - z) p + y^2 (z - x)q = z^2(x - y)$

Which is of the form Pp + Qq = R

Step-2 : Subsidiary Equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Step-3 : Here $P = x^2 (y - z)$; $Q=y^2 (z - x)$; $R = z^2 (x - y)$

Choose multiplier P', Q', R' as $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

Choose multiplier P'', Q'', R'' as $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$

Step-4 : $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

$$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = C$$

$$\Rightarrow \log x + \log y + \log z = C$$

$$\Rightarrow \log xyz = C$$

$$\Rightarrow xyz = C_1 \quad \text{...................... (1)}$$

Similarly $\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$

Integrating $\int \frac{1}{x^2}dx + \int \frac{1}{y^2}dy + \int \frac{1}{z^2}dz = k$

$$\Rightarrow -\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = k$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = k_1 \quad \text{...................... (2)}$$

Combining equation (1) and (2) we get our desire result.

$$f(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}) = 0$$

Example : Solve pyz + qzx = xy

Step-1 : Given D.E. is

Pyz + qzx = xy

Which is of the form Pp + Qq = R

Step-2 : Subsidiary equations are

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

Step-3 : Here $\frac{dx}{yz} = \frac{dy}{zx}$

$$\Rightarrow \frac{dx}{y} = \frac{dy}{x}$$

$$\Rightarrow xdx = ydy$$

On integration, this yields,

$$\int xdx = \int ydy$$

$$\Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + C$$

$$\Rightarrow \frac{x^2}{2} - \frac{y^2}{2} = C$$

$$\Rightarrow x^2 - y^2 = C_1 \text{.................. (1)}$$

Again $\frac{dy}{zx} = \frac{dz}{xy}$

$$\Rightarrow \frac{dy}{z} = \frac{dz}{y}$$

Ydy=zdz

On integration, this yields,

$$\int y\,dy = \int z\,dz$$

$$\Rightarrow \frac{y^2}{2} = \frac{z^2}{2} + k_1$$

$$\Rightarrow \frac{y^2}{2} - \frac{z^2}{2} = k_1$$

$$\Rightarrow y^2 - z^2 = k \text{ .... (2)}$$

Combining (1) and (2), we get the desire result, $f(x^2 - y^2, y^2 - z^2) = 0$

Long Type Question with answer of ODE

1. Find P.I of $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$

Solution : Given D.E. is $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$

In symbolic form $D^3y + 4Dy = \sin 2x$

$\Rightarrow (D^3 + 4D)y = \sin 2x$

P.I = $\frac{1}{D^3 + 4D}\sin 2x$

$F(D) = D^3 + 4D$, Put $D^2 = -a^2$

$F(-a^2 = D^2) = D(-2^2) + 4D = -4D + 4D = 0$

$= x\frac{1}{3D^2 + 4}\sin 2x$

$= x\frac{1}{3(-2^2) + 4}\sin 2x$

$= \frac{x \sin 2x}{-12 + 4} = \frac{x \sin 2x}{-8}$

Q2. Solve : $(D^2 + 4D + 3)y = e^{-x}\sin x + xe^{3x}$

Ans. : Given D.E is

$(D^2 + 4D + 3)y = e^{-x}\sin x + xe^{3x}$

The general solution of given D.E is

$y = y_c + y_p$

To find Y_c or C. F.

The A.E is

$D^2 + 4D + 3 = 0$

$\Rightarrow D^2 + 3D + D + 3 = 0$

$\Rightarrow D(D + 3) + 1(D + 3) = 0$

$\Rightarrow (D + 3) + (D + 1) = 0$

$\Rightarrow D = -3, -1$

Hence $y_c = c_1 e^{-3x} + c_2 e^{-x}$

To find y_p

$y_p = \frac{1}{D^2 + 4D + 3}\left[e^{-x}\sin x + xe^{3x}\right]$

$= \frac{1}{D^2 + 4D + 3}e^{-x}\sin x + \frac{1}{D^2 + 4D + 3}xe^{3x}$

$= e^{-x}\frac{1}{(D-1)^2 + 4(D-1) + 3}\sin x + e^{3x}\frac{1}{(D+3)^2 + 4(D+3) + 3}x$

$= e^{-x}\frac{1}{D^2 - 2D + 1 + 4D - 4 + 3}\sin x + e^{3x}\frac{1}{D^2 + 6D + 9 + 4D + 12 + 3}x$

$= e^{-x}\frac{1}{D^2 + 2D}\sin x + e^{3x}\frac{1}{D^2 + 10D + 24}x$

$$= e^{-x}\frac{1}{-1+2D}\sin x + e^{3x}\frac{1}{24\left[1+\frac{D^2+10D}{24}\right]}x$$

$$= e^{-x}\frac{2D+1}{(2D-1)(2D+1)}\sin x + \frac{e^{3x}}{24}\left[1+\frac{D^2+10D}{24}\right]^{-1}x$$

$$= e^{-x}\frac{2D+1}{4D^2-1}\sin x + \frac{e^{3x}}{24}\left[1-\frac{D^2+10D}{24}\right]x$$

$$= e^{-x}\frac{2D+1}{4(-1)-1}\sin x + \frac{e^{3x}}{24}[x-\frac{10}{24}]$$

$$= \frac{e^{-x}}{-5}(2D+1)\sin x + \frac{e^{3x}}{24}\left(x-\frac{5}{12}\right)$$

$$= \frac{e^{-x}}{-5}(2D\sin x + \sin x) + \frac{e^{3x}}{24}\left(x-\frac{5}{12}\right)$$

$$= \frac{e^{-x}}{-5}(2\cos x + \sin x) + \frac{e^{3x}}{24}(x-\frac{5}{12})$$

Hence the complete solution is

$$y = y_c + y_p = c_1e^{-3x} + c_2e^{-x} + \frac{e^{-x}}{-5}(2\cos x + \sin x) + \frac{e^{3x}}{24}(x-\frac{5}{12})$$

Q3. Solve $(D^2+4)y = e^x\sin^2x$

Solution : The given D.E is

$(D^2 + 4)y = e^x \sin^2x$

The general solution is

$y = y_c + y_p$

To find Y_c

The A.E.

$D^2 + 4 = 0$

$\Rightarrow D^2 = -4$

$\Rightarrow D = 0 \pm 2i$

$\therefore y_c = e^{ox}$ [c_1 cos 2x + c_1 sin 2x]

$\Rightarrow y_c = c_1 \cos 2x + c_2 \sin 2x$

$$\mathbf{y_p} = \frac{1}{D^2+4}e^x\sin^2 x = \frac{1}{D^2+4}e^x\left[\frac{1-\cos 2x}{2}\right]$$

$$= \frac{1}{2}\frac{1}{D^2+4}\left[e^x - e^x\cos 2x\right] = \frac{1}{2}\frac{1}{D^2+4}e^x - \frac{1}{2}\frac{1}{D^2+4}e^x\cos 2x$$

$$= \frac{1}{2}\frac{e^x}{1^2+4} - \frac{1}{2}e^x.\frac{1}{(D+1)^2+4}\cos 2x = \frac{e^x}{10} - \frac{e^x}{2}\frac{1}{(D^2+2D+5)}\cos 2x$$

$$= \frac{e^x}{10} - \frac{e^x}{2}\frac{1}{-4+2D+5}\cos 2x = \frac{e^x}{10} - \frac{e^x}{2}.\frac{1}{2D+1}\cos 2x$$

$$= \frac{e^x}{10} - \frac{e^x}{2}.\frac{1}{2D+1}\cos 2x = \frac{e^x}{10} - \frac{e^x}{2}.\frac{2D-1}{4D^2-1}\cos 2x$$

$$= \frac{e^x}{10} - \frac{e^x}{2} . \frac{2D-1}{4(-2^2)-1} \cos 2x \quad = \frac{e^x}{10} - \frac{e^x}{2} . \frac{2D-1}{-16-1} \cos 2x$$

$$= \frac{e^x}{10} - \frac{e^x}{2} . \frac{2D-1}{-17} \cos 2x$$

$$= \frac{e^x}{10} + \frac{e^x}{34} [2D\{\cos 2x\} - \cos 2x]$$

$$= \frac{e^x}{10} + \frac{e^x}{34} [-4 \sin 2x - \cos 2x]$$

$\therefore$ Hence the general solution is

$y = y_C + Y_P$

$$= c_1 \cos 2x + c_2 \sin 2x + \frac{e^x}{10} + \frac{e^x}{34} (-4 \sin 2x - \cos 2x)$$

Q4. Solve $(D^2+2D+2)y = \sin 2t$

Ans. Given D.E is

$(D^2+2D+2)y = \sin 2t$ ……………………… (i)

The general solutionof given D.E. is $y = y_c+y_p$

To find y_c

The A. E. of Equation (i) is

$D^2+2D+2 = 0$

$$\Rightarrow \quad D = \frac{-2 \pm \sqrt{2^2 - 4.2.1}}{2.1}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$\Rightarrow \quad D = -1 \pm i$

Hence $\quad Y_C = e^{-t}[C_1 \cos t + C_2 \sin t]$

To find y_P

$$Y_P = \frac{1}{D^2 + 2D + 2} \sin 2t = \frac{1}{-4 + 2D + 2} \sin 2t \qquad \text{(Put } D^2 = -a^2 = -4)$$

$$= \frac{1}{2D-2} \sin 2t$$

$$= \frac{1}{2} \frac{D+1}{(D-1)(D+1)} \sin 2t$$

$$= \frac{1}{2} \frac{D+1}{D^2-1} \sin 2t$$

$= \frac{1}{2}\frac{D+1}{-4-1}\sin 2t$

$= \frac{1}{2(-5)}(D+1)\sin 2t$

$= -\frac{1}{10}(D\sin 2t + \sin 2t)$ $= -\frac{1}{10}(2\cos 2t + \sin 2t)$

Hence the general solution of given D.E. is $y = y_c + y_P$

$= e^{-t}[c_1 \cos t + c_2 \sin t]$ $-\frac{1}{10}(2\cos 2t + \sin 2t)$

Q5. Solve $\frac{d^2y}{dt^2} + n^2 y = k\cos(nt+\alpha)$

Ans. Given D.E. is $\frac{d^2y}{dt^2} + n^2 y = k\cos(nt+\alpha)$ .... (i)

The general solution is $y = y_C + y_P$

Y_C The symbolic form of equation (i) is

$D^2y + n^2y = k \cos(nt+\alpha)$

$\Rightarrow$ $(D^2+n^2)y = k\cos(nt+\alpha)$

To find y_c

The A. E. is $D^2+n^2 = 0$

$\Rightarrow$ $D^2 = -n^2$

$\Rightarrow$ $D=0 \pm ni$

$\therefore$ $Y_C = e^{0t}[C_1 \cos nt + C_2 \sin nt]$ $= C_1$ cost $nt + C_2 \sin nt$

To find y_P

$Y_p = \frac{1}{D^2+n^2}k\cos(nt+\alpha)$

$= kt.\frac{1}{2D}\cos(nt+\alpha)$ $= \frac{kt}{2}\int\cos(nt+\alpha)dt$

Here $F(D) = D^2 + n^2$

Putting $D^2 = -n^2$ $F(D) = -n^2+n^2 = 0$

So $F'(D) = 2D$

$Y_P = \frac{kt}{2}\int\cos(nt+\alpha)dt$ =

$\frac{kt}{2}\frac{\sin(nt+\alpha)}{n} = \frac{kt}{2n}\sin(nt+\alpha)$

Hence the general solution is $y = y_C + y_P = C_1 \cos nt + C_2 \sin nt + \frac{kt}{2n}\sin(nt+\alpha)$

P.D. E.

Long Questions with Answer

Q1. Solve $x^2(y-z)P + y^2(z-x)q = z^2(x-y)$

Solution

$x^2(y-z)P + y^2(z-x)q = z^2(x-y)$

Given question is of the form of Pp+Qq = R

It's subsidiary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(y-x)} = \frac{dz}{z^2(x-y)}$$

Here $P = x^2(y-z)$, $Q=y^2(z-x)$, $R=z^2(x-y)$

Let $P' = \frac{1}{x}$, $Q' = \frac{1}{y}$, $R' = \frac{1}{z}$

Then $PP'+QQ'+RR'$

$= x^2(y-z)\frac{1}{x} + y^2(z-x).\frac{1}{y} + z^2(x-y).\frac{1}{z}$

$= x(y-z) + y(z-x) + z(x-y)$

$= xy - xz + yz - xy + zx - yz = 0$

Hence $P'dx + Q'dy + R'dz = 0$ is integrable

$\Rightarrow \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

On integration,

$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = \int 0$

$\Rightarrow \quad \log x + \log y + \log z = \log C_1$

$\Rightarrow \quad \log(xyz) = \log C_1$

$\Rightarrow \quad xyz = C_1$ …………………………(i)

Again let $P'' = \frac{1}{x^2}$, $Q'' = \frac{1}{y^2}$, $R'' = \frac{1}{z^2}$

Then $PP''+QQ''+RR''$

$= x^2(y-z).\frac{1}{x^2} + y^2(z-x).\frac{1}{y^2} + z^2(x-y).\frac{1}{z^2}$

$= y - z + z - x + x - y = 0$

Hence $P''dx + Q''dy + R''dz = 0$ is integrable.

$\Rightarrow \quad \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$

On integration

$\int \frac{1}{x^2}dx + \int \frac{1}{y^2}dy + \int \frac{1}{z^2}dz = \int 0$

$\Rightarrow \quad -\frac{1}{x}-\frac{1}{y}-\frac{1}{z}=C_2$

$\Rightarrow \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=C_2$ ………………………..(ii)

Hence from (1) & (2), the complete solution is $f\left(xyz, \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=0$ or $xyz=f\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

Q2 Solve $x(y^2-z^2)P+y(z^2-x^2)q-z(x^2-y^2)=0$

Ans. $x(y^2-z^2)P+y(z^2-x^2)q-z(x^2-y^2)=0$

$\Rightarrow \quad x(y^2-z^2)P+y(z^2-x^2)q=z(x^2-y^2)$

Hence $P= x(y^2-z^2)$, $Q=y(z^2-x^2)$, $R=z(x^2-y^2)$

It's subsidiary equation are

$$\frac{dx}{x(y^2-z^2)}=\frac{dy}{y(z^2-x^2)}=\frac{dz}{z(x^2-y^2)}$$

Let $P'=\frac{1}{x}, \quad Q'=\frac{1}{y}, \quad R'=\frac{1}{z}$

Then $PP'+QQ'+RR'$

$= x(y^2-z^2).\frac{1}{x}+y(z^2-x^2).\frac{1}{y}+z(x^2-y^2).\frac{1}{z}$

$= y^2 - z^2+z^2-x^2+x^2-y^2= 0$

$P'dx + Q'dy+R'dz=0$ is integrable

$\Rightarrow \quad \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$

On integration

$$\int\frac{1}{x}dx+\int\frac{1}{y}dy+\int\frac{1}{z}dz=\int 0$$

$\Rightarrow \quad \log x + \log y + \log z = C_1$

$\Rightarrow \quad \log (xyz)=\log c_1$

$\Rightarrow \quad xyz = C_1$ ……............... (i)

Again let $P''=x, \quad Q''=y \quad R''=z$

Then $PP''+QQ''+RR''$

$= x . x (y^2 - z^2) +y.y(z^2- x^2) +z.z(x^2-y^2)$

$= x^2 (y^2 - z^2) +y^2(z^2- x^2) +z^2(x^2-y^2)$

$= x^2y^2-x^2z^2+y^2z^2-x^2y^2 + z^2x^2-y^2z^2 = 0$

Hence $P''dx+Q''dy+R''dz = 0$ is integrable

$\Rightarrow \quad xdx + ydy + zdz = 0$

On integration,

$$\int xdx + \int ydy + \int zdz = \int 0$$

$$\Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_2$$

$$\Rightarrow \quad \frac{x^2 + y^2 + z^2}{2} = c_2$$

From (i) & (ii), the complete solution is $f\left(xyz, \frac{x^2 + y^2 + z^2}{2}\right) = 0$ or $xyz = f\left(\frac{x^2 + y^2 + z^2}{2}\right)$

Q3 Solve $(x^2 - yz)P + (y^2 - zx)q = z^2 - xy$

Solution :

$(x^2 - yz)P + (y^2 - zx)q = z^2 - xy$

Here $P = x^2 - yz$, $Q = y^2 - zx$, $R = z^2 - xy$

It's subsidiary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

Let $P' = y$, $Q' = z$, $R' = x$

Then $PP' + QQ' + RR'$

$= y.(x^2 - yz) + z(y^2 - zx) + x(z^2 - xy)$

$= x^2 y - y^2 z + y^2 z - z^2 x + z^2 x - x^2 y = 0$

Hence $P' dx + Q' dy + R' dz = 0$ is integrable

$\Rightarrow$ $ydx + ydy + xdz = 0$

On integration

$$\int ydx + \int zdy + \int xdz = \int 0$$

$\Rightarrow$ $xy + yz + xz = C_1$ ………………………. (i)

Again considering the 1st and last two ratio

$$\frac{dx - dy}{x^2 - yz - y^2 + zx} = \frac{dy - dz}{y^2 - zx - z^2 + xy}$$

$$\Rightarrow \quad \frac{d(x - dy)}{x^2 - y^2 + zx - yz} = \frac{dy - dz}{y^2 - z^2 + xy - zx}$$

$$\Rightarrow \quad \frac{dx - dy}{(x - y)(x + y) + z(x - y)} = \frac{dy - dz}{(y - z)(y + z) + x(y - z)}$$

$$\Rightarrow \quad \frac{dx - dy}{(x - y)(x + y + z)} = \frac{dy - dz}{(y - z)(x + y + z)}$$

$$\Rightarrow \quad \frac{d(x - y)}{x - y} = \frac{d(y - z)}{y - z}$$

On Integrating

$$\int \frac{d(x - y)}{x - y} = \int \frac{d(y - z)}{y - z}$$

$\Rightarrow$ $\log(x-y) = \log(y-z) + \log C_2$

$\Rightarrow$ Log(x-y)-log(y-z) = log C_2

$\Rightarrow$ $\log\left(\dfrac{x-y}{y-z}\right) = \log C_2$

$\Rightarrow$ $\dfrac{x-y}{y-z} = C_2$ ………………………….. (ii)

Hence from (i) & (ii), the complete solution is $f\left(xy+yz+zx, \dfrac{x-y}{y-z}\right) = 0$

Or $xy+yz+zx = f\left(\dfrac{x-y}{y-z}\right)$

Q4. Solve 2P-q = Z

Solution

2p –q=z

Hence P = 2, Q=-1, R=z

It's subsidiary equations are

$$\frac{dx}{2} = \frac{dy}{-1} = \frac{dz}{z}$$

Consider the 1st two ratios

$$\frac{dx}{2} = \frac{dy}{-1}$$

$\Rightarrow$ - dx = 2dy

On integration

$$\int -dx = \int 2dy$$

$\Rightarrow$ -x=2y+c $\Rightarrow$ 2y+x=C_1 ………………..(i)

$$\frac{dy}{-1} = \frac{dz}{z}$$

$\Rightarrow$ $\int -dy = \int \dfrac{dz}{z}$ $\Rightarrow$ -y = logz + c_2

$\Rightarrow$ log z+y = C_2 …………………………… (ii)

$\therefore$ From (i) & (ii) the complete solution is f(2y+x, log z+y) = 0

Q5. Solve xp-yq = y^2 –x^2

Solution

xp-yq = y^2 –x^2

Hence P=x, Q= -y, R=y^2-x^2

It's subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2 - x^2}$$

Consider the 1st two ratios

$$\frac{dx}{x} = \frac{dy}{-y}$$

On integration,

$$\int \frac{dx}{x} = \int \frac{dy}{-y}$$

$\Rightarrow$ $\log x = -\log y + \log c_1$

$\Rightarrow$ $\log x + \log y = \log c_1$

$\Rightarrow$ $\log(xy) = \log c_1$

$\Rightarrow$ $xy = C_1$ ……………………….. (i)

Again, Consider the three ratios

$$\frac{dx - dy}{x - (-y)} = \frac{dz}{y^2 - x^2}$$

$$\Rightarrow \frac{dx - dy}{x + y} = \frac{dz}{(y - x)(y + x)}$$

$\Rightarrow$ $(y - x)(dx - dy) = dz$

$\Rightarrow$ $ydx - ydy - xdx + xdy = dz$

$\Rightarrow$ $ydx + xdy - ydy - xdx = dz$

$\Rightarrow$ $d(xy) - ydy - xdx - dz = 0$

On integration,

$$\int d(xy) - \int xdx - \int ydy - \int dz = \int 0$$

$$\Rightarrow xy - \frac{x^2}{2} - \frac{y^2}{2} - z = c_2$$

$$\Rightarrow \frac{2xy - x^2 - y^2 - 2z}{2} c_2$$

$$\Rightarrow -(x^2 + y^2 - 2xy) - 2z = 2c_2$$

$\Rightarrow$ $-\{(x - y)^2 - 2z\} = 2c_2$ $\Rightarrow$ $(x-y)^2+2z=c_3$ …………………… (ii)

Hence from (i) & (ii), the complete solution is

$F(xy, (x-y)^2+2z)=c_2$

Or $xy = f\{(x-y)^2 + 2z\}$

Q6. Solve x(y-z)p + y(z-x)q = z(x-y)

Solution :

$x(y-z)P+y(z-x)q = z(x-y)$

Here $P = x(y-z)$, $Q = y(z-x)$, $R=z(x-y)$

It's subsidiary equations are $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$

Let $P' = 1,\ Q' = 1,\ R' = 1$

Then $PP' + QQ' + RR'$

$= x(y-z).1 + y(z-x) . 1 + z(x - y). 1$

$= xy - xz + yz - yx + zx - yz = 0$

Hence $P'dx + Q'dy + R'dz = 0$ is integrable.

$\Rightarrow \quad 1dx + 1.dy + 1.dz = 0 \qquad \Rightarrow \quad dx + dy + dz = 0$

On integration,

$$\int dx + \int dy + \int dz = \int 0$$

$\Rightarrow \quad x+y+z = C_1$ ……………… (i)

Again let $P'' = \frac{1}{x}$, $Q'' = \frac{1}{y}$, $R'' = \frac{1}{z}$

Then $PP'' + QQ'' + RR''$

$= x(y-z).\frac{1}{x} + y(z-x).\frac{1}{x} + z(x-y).\frac{1}{z}$

$= y - z + z - x + x - y = 0$

Hence $P''dx + Q''dy + R''dz = 0$ is integrable.

$\Rightarrow \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

On integration

$$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = \int 0$$

$\Rightarrow \quad$ Log x + log y + log z = log C_2

$\Rightarrow \quad$ log (xyz) = log C_2

$\Rightarrow \quad xyz = C_2$ ………………… (ii)

Hence from (1) & (2), the complete solution is $f(x+y+z, xyz) = 0$

Or $x + y + z = f(xyz)$

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Chapter - 4
Laplace Transform

4.1 Gamma Function

The gamma function is defined as

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx \qquad (n>0)$$

In particular, $\Gamma(1)=\int_0^\infty e^{-x}dx=\left[\frac{e^{-x}}{-1}\right]_0^\infty=(-1)\left[\frac{1}{e^\infty}-\frac{1}{e^0}\right]=(-1)(0-1)=1$

Reduction formula for $\Gamma(n)$ **: :** $\Gamma(n+1)=n\Gamma(n)$

Proof : Since $\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx, \qquad (n>0)$

$\therefore \quad \Gamma(n+1)=\int_0^\infty e^{-x}.x^{n+1-1}dx$ [putting (n+1) in place of n]

$$=\int_0^\infty e^{-x}.x^n dx$$

$$=x^n(-e^{-x})]_0^\infty-\int_0^\infty nx^{n-1}.(-e^{-x})dx$$

$$=0+n\int_0^\infty e^{-x}.x^{n-1}.dx \qquad \left(\lim_{n\to\infty}\left(\frac{1}{n}\right)=0\right)$$

$$=n\Gamma(n)$$

we know $\Gamma(1)=1$

Using $\Gamma(n+1)=n\Gamma(n)$ successively, we get

$\Gamma(2)=\Gamma(1+1)=1.\Gamma(1)=1.1=1!$

$\Gamma(3)=\Gamma(2+1)=2.\Gamma(2)=2.1=2!$

....

....

....

In general

$\Gamma(n+1)=n\Gamma(n)=n(n-1)!=n!$

$\Rightarrow \quad \Gamma(n+1)=n!$, provided n is a positive integer

Taking n=0, $\Gamma(0+1)=0!$

but $\Gamma(1)=1 \qquad (\ 0! = 1)$

When n is a fraction $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi},\ \Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi},$

4.2 Laplace Transforms

Laplace transform operator 'L' is simillar to differential operator 'D'. Both are linear in nature so they obey linearity property where L is operated on f(t), the result is function of 's'. i.e. f(s)

Introduction :

The method of laplace transforms has the advantage of directly giving the solution of differential equation (both for homogeneous / nonhomogeneous) with given boundary values without finding the general solution.

Definition :

Let f(t) be a function of 't' defined for all positive values of 't'. Then the laplace transforms of f(t), denoted by L{f(t)} is defined by

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$, Provided that the integral exists.

Here S is a parameter which may be a real or complex number.

L{f(t)} being a function of s written as

$$L\{f(t)\} = \bar{f}(s)$$

$$\Rightarrow \quad f(t) = L^{-1}\{\bar{f}(s)\}$$

Here $L \rightarrow$ laplace transformation operator.

$L_{-1} \rightarrow$ inverse laplace transformation operator.

4.3 Condition for existence

The laplace transform of f(t) i.e. $\int_0^\infty e^{-st} f(t)dt$ exists for s > a, if

i) f(t) is continuous ii) $\lim_{t\to\infty}\{e^{-at} f(t)\}$ is finite

Laplace Transform of standard functions (Memorised the following formula)

1) $L\{1\} = \frac{1}{s}$ 2) $L\{e^{at}\} = \frac{1}{s-a}$

3) $L\{t^n\} = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{S^{n+1}}$ 4) $L\{\sin at\} = \frac{a}{s^2+a^2}$

5) $L\{\sin h\, at\} = \frac{a}{s^2-a^2}$ 6) $L\{\cos at\} = \frac{s}{s^2+a^2}$

7) $L\{\cos h\, at\} = \frac{s}{s^2-a^2}$

1) $L\{1\} = \frac{1}{s}$

Proof : We know $L\{f(t)\} = \int_0^\infty e^{-st}.f(t)dt$

$$\Rightarrow L(1) = \int_0^\infty e^{-st} x1.dt = \left.\frac{e^{-st}}{-s}\right]_0^\infty = \frac{-1}{s}\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right] = -\frac{1}{s}[0-1] = \frac{1}{S}$$

2) $L\{e^{at}\} = \frac{1}{s-a}$

Proof $L\{e^{at}\} = \int_0^\infty e^{-st}.e^{at}dt = \int_0^\infty e^{-(s-a)t}dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{-1}{(s-a)}(0-1) = \frac{1}{s-a}$, $s > a$

Note : L. T doesn't exist when S < a

3) $L\{t^n\} = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{S^{n+1}}$

Proof $L\{t^n\} = \int_0^\infty e^{-st}.t^n dt = \int_0^\infty e^{-p}.\left(\frac{p}{s}\right)^n \frac{dp}{s}$ Put st = p ⇒ S= $\frac{dp}{dt} \Rightarrow dt = \frac{dp}{s}$

$= \frac{1}{s^{n+1}}\int_0^\infty e^{-p}.p^n dp = \frac{\Gamma(n+1)}{S^{n+1}}$ ⇒ $t = \frac{P}{S}$

i.e $L(t^n) = \frac{n!}{s^{n+1}}$ (when n is a +ve integer)

in particular $L\left(t^{-\frac{1}{2}}\right) = \frac{\Gamma(1/2)}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{s^{\frac{1}{2}}} = \sqrt{\frac{\pi}{s}}$

$L\left(t^{\frac{1}{2}}\right) = \frac{\Gamma(1/2+1)}{s^{\frac{1}{2}+1}} = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$

4) L {sin at} = $\int_0^\infty e^{-st}.\sin at.dt = \left[\frac{e^{-st}}{(-s)^2+a^2}\left(-s\sin at - a\cos at\right)\right]_0^\infty = \frac{-1}{s^2+a^2}(0-a) = \frac{a}{s^2+a^2}$

7) L {cos h at} = $\int_0^\infty e^{-st}.\cosh at.dt = \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right)dt$

$= \frac{1}{2}\int_0^\infty (e^{-st}.e^{at} + e^{-st}.e^{-at})dt = \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}dt + \int_0^\infty e^{-(s+a)t}dt\right]$

$= \frac{1}{2}\left[\frac{e^{-(s-a)}t}{-(s-a)} + \frac{e^{-(s+a)}t}{-(s+a)}\right]_0^\infty = \frac{1}{2}\left[0-(-1)\left(\frac{1}{(s-a)}+\frac{1}{(s+a)}\right)\right]$

$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{1}{2}.\frac{2s}{s^2-a^2} = \frac{s}{s^2-a^2}$

Similarily we can prove

5) L(cos at) = $\frac{s}{s^2+a^2}$

6) L (sin h at) = $\frac{a}{s^2-a^2}$

Example : 1

Find the laplace transform of

i) cos2t ii) sin 4t iii)t_4 iv) e_{2t}

Sol. i) L{cos2t} = $\frac{s}{s^2+2^2} = \frac{s}{s^2+4}$

ii) L{sin4t} = $\frac{4}{s^2+4^2} = \frac{4}{s^2+16}$

iii) L{t_4} = $\frac{4!}{s^{4+1}} = \frac{4x3x2x1}{s^5} = \frac{24}{s^5}$

iv) L{e_{2t}} = $\frac{1}{s-2}$

4.4 Properties of Laplace transforms :

I) **Linearity property :** if a, b, c be any constants and f, g, h any function of t, then

L {af(t)+bg(t)-ch(t)} = a L{f(t)}+bL{g(t)}-cL{h(t)}

Example :

Find L{e_{2t} + 4t_3 - 2sin3t + 3cos3t}

Sol. L{e_{2t} + 4t_3 - 2sin3t + 3cos3t}

= L{e_{2t}}+L{4t_3}-L{2sin3t}+L{3cos3t}

= L {e_{2t}} +4L{t_3} - 2L{sin3t}+3L{cos3t}

= $\frac{1}{s-2} + 4.\frac{3!}{s^{3+1}} - 2.\frac{3}{s^2+3^2} + 3.\frac{s}{s^2+3^2}$

= $\frac{1}{s-2} + \frac{24}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+9}$

II) **First shifting property :**

If L{f(t)} = $\bar{f}(s)$,

then $L\{e^{at}f(t)\} = \bar{f}(s-a)$

Proof $L\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt$

= $\int_0^\infty e^{-rt}f(t)dt,\ \text{where } r = s-a$

= $\bar{f}(r) = \bar{f}(s-a)$

Thus Ipif we know the transform of f(t), we can write the transform of e_{at} f(t) simply replacing s by (s-a) to get i.e. if

$\Rightarrow$ L{e_{at}. f(t)}=(s-a)

Application of 1st shifting property

1) $L\{1\} = \frac{1}{s}$ $L\{e^{at}\} = \frac{1}{s-a}$

2) $L\{t^n\} = \frac{n!}{s^{n+1}}$ $L\{e^{at}.t^n\} = \frac{n!}{(s-a)^{n+1}}$

3) $L\{\sin bt\} = \frac{b}{s^2+b^2}$ $L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$

4) $L\{\cos bt\} = \frac{s}{s^2+b^2}$ $L\{e^{at}\cos bt\} = \frac{(s-a)}{(s-a)^2+b^2}$

5) $L\{\sinh bt\} = \frac{b}{s^2 - b^2}$ $\qquad L\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2 - b^2}$

6) $L\{\cosh bt\} = \frac{s}{s^2 - b^2}$ $\qquad L\{e^{at}\cosh bt\} = \frac{(s-a)}{(s-a)^2 - b^2}$

Example : Find the laplace transforms of

$e_{-3t}(2\cos 5t - 3\sin 5t)$.

Sol. L $\{e_{-3t}(2\cos 5t - 3\sin 5t)\}$.

$= L\{e^{-3t}2\cos 5t\} - L\{e^{-3t}3\sin 5t\}$

$= 2L\{e^{-3t}\cos 5t\} - 3L\{e^{-3t}\sin 5t\}$

$= 2.\frac{s+3}{(s+3)^2 + 5^2} - 3.\frac{5}{(s+3)^2 + 5^2} = \frac{2s-9}{s^2 + 6s + 34}$

Example : Find the Laplace transform of

$$F(t) = \begin{cases} 1, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ 0, & t > 2 \end{cases} \qquad (t > 2 \text{ i.e. } 2 \text{ to } \infty)$$

Sol. $L\{F(t)\} = \int_0^\infty e^{-st}.f(t)dt$

$= \int_0^1 e^{-st}.1.dt + \int_1^2 e^{-st}.t.dt + \int_2^\infty e^{-st}.0.dt$

$= \left|\frac{e^{-st}}{-s}\right|_0^1 + \left|t.\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_1^2$

$= \frac{1-e^{-s}}{s} + \left\{\left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2}\right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right)\right\}$

$= \frac{1}{s} - \frac{e^{-s}}{s} - \frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2} = \frac{1}{s} - \frac{2e^{-2s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}$

III. Change of scale property:

If $\quad$ L{f(t)} = then L {f(at)} = $\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

Example

Applying Change of scale property

Find L{at}

Sol. We know $L\{t\} = \frac{1}{s^2} = \bar{f}(s)$

$L\{at\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right) = \frac{1}{a}.\frac{1}{\left(\frac{s}{a}\right)^2} = \frac{1}{a} x \frac{a^2}{s^2} = \frac{a}{s^2}$

By using change scale property

Laplace Transforms of function $L\{t^n f(t)\}$ and $L\left\{\frac{1}{t}f(t)\right\}$

4.5(a) Multiplication by t_n

If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n[\bar{f}(s)]}{ds^n}$$, where n = 1, 2, 3, ...

Example

Find the Laplace transforms of t cos at

Solution

Since L{cos at} = $s/(s_2+a_2)$

$\therefore$ L{t cos at } = $(-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = -\left[\frac{(s^2+a^2)-s.2s}{(s^2+a^2)^2}\right] = \frac{-(s^2-a^2)^2}{(s^2+a^2)^2}$

Example

Find the Laplace transforms of t_2 sin at

Since L (Sin at)= $\frac{a}{s^2+a^2}$, = $L(t\sin at) = (-1)^1\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = (-1).\left\{\frac{(s^2+a^2).0-a.2s}{(s^2+a^2)^2}\right\}$ = $\frac{2as}{(s^2+a^2)^2}$

L(t_2 sin at) = $L(t.t\sin at) = (-1)^1\frac{d}{ds}\left\{\frac{2as}{(s^2+a^2)^2}\right\}$

$= -2a\left\{\frac{(s^2+a^2)^2.1 - s.2(s^2+a^2).2s}{(s^2+a^2)^4}\right\}$

$= -2a.\frac{(s^2+a^2)(s^2+a^2-4s^2)}{(s^2+a^2)^4}$

$= \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$

b) Division by t

If $L\{f(t)\} = \bar{f}(s),$ then $L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty \bar{f}(s)ds$ provided the integral exists.

Example

Find the $L\left\{\frac{\cos at - \cos bt}{t}\right\}$

Solution

Since L {cos at - cos bt}

= L{cos at} - L {cos bt}

$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$\therefore$ $L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds = \left|\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right|_s^\infty$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b}\right)\right]_s^\infty = \frac{1}{2}\left[\log\frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{1+0}{1+0}\right)-\log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)\right] = -\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) \qquad \left[\log\frac{a}{b} = -\log\frac{b}{a}\right]$$

$$= \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{\frac{1}{2}}$$

c) Transforms of Derivatives

If $f'(t)$ be continuous and $L\{f(t)\} = \bar{f}(s)$

Then $L\{f'(t)\} = s\bar{f}(s) - f(0)$ ....(1)

By applying equation (1) to the second derivative $f''(t)$ we obtain

$L(f'') = SL\{f'(t)\} - f'(0)$

$= s(s\bar{f}(s) - f(0)) - f'(0)$

$= s^2\bar{f}(s) - sf(0) - f'(0)$

$= s^2L(f(t)) - sf(0) - f'(0)$ ....(2)

$\Rightarrow$ $L\{f'''(t)\} = s^2\bar{f}(s) - s^2f(0) - sf'(0) - f''(0)$ .... (3)

Simillary in general

$L\{f^n(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots\ldots f^{n-1}(0)$

d) Transforms of integrals

If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\bar{f}(s)$

Example :

Find L. T. of $\int_0^t e^{-t}.\cos t \; dt$

Solution : $L\{e^{-t}\cos t\} = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2}$

$\therefore$ $L\int_0^t e^{-t}\cos t \; dt = \frac{1}{s}L\{e^{-t}\cos t\} = \frac{1}{s}\cdot\frac{s+1}{s^2+2s+2}$

4.6 Inverse Laplace transforms

If $L\{f(t)\} = \bar{f}(s) \Rightarrow L^{-1}\{\bar{f}(s)\} = f(t)$

1) $L^{-1}\left[\frac{1}{s}\right]=1$

2) $L^{-1}\left[\frac{1}{s-a}\right]=e^{at}$

3) $L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{(n-1)!}$, n=1, 2, 3, ...

4) $L^{-1}\left[\frac{1}{(s-a)^n}\right]=\frac{e^{at}t^{n-1}}{(n-1)!}$

5) $L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{1}{a}\sin at$

6) $L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at$

7) $L^{-1}\left(\frac{1}{s^2-a^2}\right)=\frac{1}{a}\sinh at$

8) $L^{-1}\left(\frac{s}{s^2-a^2}\right)=\cosh at$

9) $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right]=\frac{1}{b}e^{at}\sin bt$

10) $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right]=e^{at}\cos bt$

11) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]=\frac{1}{2a}t\sin at$

12) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]=\frac{1}{2a^3}(\sin at-at\cos at)$

Example :

1. $L^{-1}\left(\frac{1}{s^2+4}\right)=\frac{1}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right)=\frac{1}{2}.\sin 2t$

2. $L^{-1}\left(\frac{s}{s^2+9}\right)=L^{-1}\left(\frac{s}{s^2+3^2}\right)=\cos 3t$

3. $L^{-1}\left(\frac{1}{(s-3)^2+4}\right)=\frac{1}{2}L^{-1}\left(\frac{2}{(s-3)^2+2^2}\right)=\frac{1}{2}e^{3t}.\sin 2t$

4. $L^{-1}\left(\frac{s}{(s+3)^2+4}\right)=L^{-1}\left(\frac{s}{(s+3)^2+2^2}\right)=e^{-3t}.\cos 2t$

Example : find the inverse Laplace transforms of $\frac{s^2-3s+4}{s^3}$

Solution :

$=L^{-1}\left(\frac{s^2-3s+4}{s^3}\right)$

$=L^{-1}\left(\frac{s^2}{s^3}-\frac{3s}{s^3}+\frac{4}{s^3}\right)$

$=L^{-1}\left(\frac{1}{s}-\frac{3}{s^2}+\frac{4}{s^3}\right)$

$=L^{-1}\left(\frac{1}{s}\right)-3\,L^{-1}\left(\frac{1}{s^2}\right)+4.L^{-1}\left(\frac{1}{s^3}\right)$

$= 1 - 3t + 2t_2$

To solve a inverse Laplace transform we use the partial fraction method :-

Algebraic fraction is of two types.

1) Proper fraction
2) Improper fraction.

(1) Proper Fraction :

If the degree of the Numerator is less than the degree of the denominator is known as proper fraction.

For e.g. $\frac{1}{(s+1)(s+2)}$, $\frac{2s}{s^2+3s+2}$ and $\frac{s^2}{(s-1)(s-2)(s-3)}$ etc.

2. Improper fraction :

If the degree of the numerator is greater than or equal to the degree of denominator, the fraction is called improper fraction.

For eg. $\frac{s^2-s+2}{s+1}$, $\frac{s^5-7s^2+2s+3}{s^2+s+3}$ etc. $N_0 \geq D_0$

When we use improper fraction dividing numerator by denominator and express the function as a sum of polynomial function and a proper fraction which is obtained from Dividend = quotient + $\frac{\text{remainder}}{\text{divisor}}$

for eg. $\frac{x^3-1}{x^3+1}=1-\frac{2}{x^3+1}$

For proper fraction, where denominator are of linear nonrepeated, Linear repeated, Quadratic non repeated and quadratic repeated such as :

There are 4 cases arises

a) Non - repeated linear factor (case - a)

b) Repeated linear factor (case - b)

c) Non repeated quadratic factor (case - c)

d) Repeated quadratic factor. (case - d)

		Form of partial Fraction
Case - a	(Non-repeated linear Factor $\frac{ps+q}{(s-a)(s-b)}$	$\frac{A}{s-a}+\frac{B}{s-b}$
Case - b	(Repeated linear factor) $\frac{ps+q}{(s-a)^2}$	$\frac{A}{s-a}+\frac{B}{(s-a)^2}$
Case - c	$\frac{PS+q}{(as^2+bs+c)(ps^2+qs+r)}$	$\frac{As+B}{as^2+bs+c}+\frac{Cs+D}{ps^2+qs+r}$
Case - d	$\frac{ps+q}{(as^2+bs+c)^2}$	$\frac{As+B}{as^2+bs+c}+\frac{Cs+D}{(as^2+bs+c)^2}$

Solve linear differential equation with constant coefficients associated with initial conditions using Transforms method (upto 2nd order)

Working Procedure

To solve a linear differential equation with constant coefficients by transform method.

Step - 1 Taking laplace transform on both side of given differential equation

Step - 2 Apply given initial condition to reduce the equation

Step - 3 Solve the reduced equation by partial fraction

Step - 4 Taking inverse laplace transform on both sides to obtained the desired result.

Example -1

Solve $y''-4y'+3y = 6t-8, \quad y(0)=0, \; y'(0)=0$

Solution : Taking laplace transform both sides.

$L\{y''-4y'+3y\} = L\{6t-8\}$

$\Rightarrow \; L\{y''\} - L\{4y'\} + L\{3y\} = L\{6t\} - L\{8\}$

$\Rightarrow \; s^2\bar{y}(s) - sy(0) - y'(0) - 4\{s\bar{y}(s) - y(0)\} + 3\bar{y}(s) = \frac{6}{s^2} - \frac{8}{s}$

$\Rightarrow \; s^2\bar{y}(s) - 4s\bar{y}(s) + 3\bar{y}(s) - s.0 - 0 = \frac{6}{s^2} - \frac{8}{s}$

$\Rightarrow \; \bar{y}(s)(s^2-4s+3) = \frac{6-8s}{s^2}$

$\Rightarrow \; \bar{y}(s) = \frac{6-8s}{s^2} \, x \, \frac{1}{s^2-4s+3}$

$\Rightarrow \; L\{y\} = \frac{6-8s}{s^2(s-3)(s-1)}$

$\Rightarrow \; y = L^{-1}\left\{\frac{6-8s}{s^2(s-3)(s-1)}\right\}$

$\Rightarrow \; y = L^{-1}\left\{\frac{6-8s}{s^2(s-3)(s-1)}\right\}$

Now $\frac{6-8s}{s^2(s-3)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{s-1}$

$= \frac{As(s-3)(s-1) + B(s-3)(s-1) + Cs^2(s-1) + Ds^2(s-3)}{s^2(s-3)(s-1)}$

$\Rightarrow$ 6-8s = As(s-3)(s-1)+B(s-3)(s-1)+Cs_2(s-1)+Ds_2(s-3) ....(1)

Put (s-1)= 0S=1 is equation(1) we get.

6-8.1=A.0 +B.0 + C.0 + D.1(-2)

$\Rightarrow$ -2 = -2D $\Rightarrow$ D=1

Put s-3 = 0$\Rightarrow$ s=3 in equation (1) we get

$\Rightarrow$ 6-8.3 = A.0 + B.0 + C.9.2 + D.0

$\Rightarrow$ -18 = 18C

$\Rightarrow$ C = -1

Put s = 0 in equation (1) we get

6 - 3.0 = A.0 + B(-3)(-1)+ C. 0 + D.0

$\Rightarrow$ 6 = 3B $\Rightarrow$ B = 2

Put S = -1 in equation (1) we get

6-8 . (-1) = A. (-1)(-4)(-2) + B. (-4)(-2)+C.1.(-2)+D.1(-4)

14 = -8A + 8B - 2C - 4D

= -8A + 8.2 - 2.(-1)-4.1

8A = 16+2-4-14

A = 0

$$\therefore \quad y = L^{-1}\left(\frac{0}{s} + \frac{2}{s^2} - \frac{1}{s-3} + \frac{1}{s-1}\right)$$

$$= L^{-1}(0) + L^{-1}\left(\frac{2}{s^2}\right) - L^{-1}\left(\frac{1}{s-3}\right) + L^{-1}\left(\frac{1}{S-1}\right)$$

= 0+2t - e_{3t}+e_t

$$\therefore \quad y(t) = 2t - e^{3t} + e^t$$

Example : 2

Use transform method to solve.

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t \text{ with } x=2, \ \frac{dx}{dt} = -1, \ t = 0$$

Solution : Taking laplace transform both sides

$$L\left\{\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x\right\} = L\{e^t\}$$

$$\Rightarrow \quad L\{x''-2x'+x\} = \frac{1}{s-1}$$

$$\Rightarrow \quad L\{x''\} - L\{2x'\} + L\{x\} = \frac{1}{s-1}$$

$$\Rightarrow \quad s_2 \ \bar{x}(s)\text{-}sx(0)\text{-}x^1(0) - 2[s\bar{x}(s) - x(0)] + \bar{x}(s) = \frac{1}{s-1}$$

$$\Rightarrow \quad s^2\bar{x}(s) - s.2 - (-1) - 2[s.\bar{x}(s) - 2] + \bar{x}(s) = \frac{1}{s-1}$$

$$\Rightarrow \quad s^2\bar{x}(s) - 2s + 1 - 2s\bar{x}(s) + 4 + \bar{x}(s) = \frac{1}{s-1}$$

$$\Rightarrow \quad (s^2 - 2s + 1)\bar{x}(s) = \frac{1}{s-1} + 2s - 5$$

$$\Rightarrow \quad (s^2 - 2s + 1)\bar{x}(s) = \frac{1 + (2s-5)(s-1)}{s-1}$$

$$\Rightarrow \quad \bar{x}(s) = \frac{1 + 2s^2 - 2s - 5s + 5}{(s-1)(s-1)^2}$$

$$\Rightarrow \quad L\{\bar{x}(s)\} = \frac{2s^2 - 7s + 6}{(s-1)^3}$$

$$\Rightarrow \quad \bar{x}(s) = L^{-1}\left\{\frac{2s^2 - 7s + 6}{(s-1)^3}\right\}$$

Now $\frac{2s^2-7s+6}{(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}$

$\Rightarrow$ $\frac{2s^2-7s+6}{(s-1)^3} = \frac{A(s-1)^2+B(s-1)+C}{(s-1)^3}$

$\Rightarrow$ $2s_2$-7s+6 = A(s-1)$_2$ + B(s-1)+C .... (1)

Put s-1=0 s=1 in equation (1) we get

2.1_2 - 7.1 + 6 = C

$\Rightarrow$ 1 = C

Again put s=0 in equation (1), we get

6 = A -B+C $\qquad$ $C = 1$

$\Rightarrow$ 6 = A-B+1

$\Rightarrow$ A-B = 5 .... (2)

Again put s=-1 in equation(1) we get

2+7+6=A.4-2B+C $\qquad$ $C = 1$

$\Rightarrow$ 15 = 4A-2B+1

$\Rightarrow$ 4A-2B = 14 ....(3)

from euqtion (2) & (3) we get

4A-2B = 14

equation (2) x 5 4A - 4B=20

– + –

2B = -6

$\Rightarrow$ B=-3

$\therefore$ A-B = 5

$\Rightarrow$ A-(-3)=5

$\Rightarrow$ A=5-3=2

$\therefore$A = 2

$\therefore$ x(t) = $L^{-1}\left\{\frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}\right\}$

= $L^{-1}\left\{\frac{2}{s-1}\right\} - 3L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + L^{-1}\left\{\frac{1}{(s-1)^3}\right\}$

= $2e_t$ - 3t . e_t + e_t. $\frac{t^2}{2!}$ =$2e_t$-3te_t+½ . e_t.t_2

Some special technique for finding Inverse Transform

We have seen that the most effective method of finding the inverse transforms is by means of partial fraction. However, various other methods are available which depend on the following important inversion formulae.

Property I.

Shifting property for Inverse laplace transform :

If $L^{-1}[\bar{f}(s)] = f(t)$, then $L^{-1}[\bar{f}(s-a)] = e^{at}f(t) = e^{at}L^{-1}[\bar{f}(s)]$

Ex. $L^{-1}\left[\dfrac{1}{(s-2)^2+1}\right] = e^{2t}\sin t$

Property II.

If $L^{-1}[\bar{f}(s)] = f(t)$, then t. $f(t) = L^{-1}\left\{-\dfrac{d}{ds}\bar{f}(s)\right\}$

Example : Find the inverse Laplace Transform of

1. $\log\left(\dfrac{s+1}{s-1}\right)$ 2) $\log\left(\dfrac{s^2+1}{s(s+1)}\right)$ 3) $\cot^{-1}\left(\dfrac{S}{2}\right)$ 4) $\tan^{-1}\left(\dfrac{2}{s}\right)$

5. $\tan^{-1}\left(\dfrac{2}{s^2}\right)$ 6) $\tan^{-1}\left(\dfrac{s}{2}\right)$ 7) $\log\left(\dfrac{1+s}{s}\right)$ 8) $\log\left(\dfrac{s+1}{(s+2)(s+3)}\right)$

9. $\cot_{-1}(s)$ 10) $\dfrac{s}{(s^2+a^2)^2}$

Solution :

1) $L^{-1}\left[\log\left(\dfrac{s+1}{s-1}\right)\right]$

Let f(t) = $L^{-1}\left[\log\left(\dfrac{s+1}{s-1}\right)\right]$

t. f(t)= $L^{-1}\left[-\dfrac{d}{ds}\bar{f}(s)\right]$ = $L^{-1}\left[-\dfrac{d}{ds}\log\left(\dfrac{s+1}{s-1}\right)\right]$ = $-L^{-1}\left[\dfrac{d}{ds}\{\log(s+1)-\log(s-1)\}\right]$

= $-L^{-1}\left[\dfrac{1}{s+1}-\dfrac{1}{s-1}\right]$ = $L^{-1}\left(\dfrac{1}{s-1}\right)-L^{-1}\left(\dfrac{1}{s+1}\right) = e^t - e^{-t}$

t x f(t) = 2 sin ht ($\sin ht = \dfrac{e^t - e^{-t}}{2}$)

$f(t) = \dfrac{2\sinh t}{t}$

3. $L^{-1}\left[\cot^{-1}\left(\dfrac{s}{2}\right)\right]$

Solution : Let f(t) = $L^{-1}\left[\cot^{-1}\left(\dfrac{s}{2}\right)\right]$

Using the above property

t.f(t) = $L^{-1}\left[-\dfrac{d}{ds}\bar{f}(s)\right]$

= $L^{-1}\left[-\dfrac{d}{ds}\left\{\cot^{-1}\left(\dfrac{s}{2}\right)\right\}\right]$

$$= -L^{-1}\left[\frac{-1}{\left(\frac{s}{2}\right)^2+1} \times \frac{1}{2}\right] = -L^{-1}\left[\frac{-2^2}{s^2+2^2} \times \frac{1}{2}\right] = L^{-1}\left[\frac{2}{s^2+2^2}\right] = \sin 2t$$

$$\therefore \quad f(t) = \frac{\sin 2t}{2}$$

10) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Solution :

Let f(t) = $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

so that $\bar{f}(s) = \frac{1}{s^2+a^2}$

Using the above property

t x f(t) = $L^{-1}\left[-\frac{d}{ds}\bar{f}(s)\right]$

$$= L^{-1}\left[-\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right] = L^{-1}\left[(-1).\left(\frac{-1}{(s^2+a^2)^2}\right) x2s\right]$$

$$\Rightarrow \quad t \times \frac{\sin at}{a} = L^{-1}\left[\frac{2s}{(s^2+a^2)^2}\right]$$

$$\Rightarrow \quad L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a} t \sin at$$

Rest problem will be solve in assignment

Property III.

If $L^{-1}\left[\bar{f}(s)\right] = f(t)$ and f(0)=0, then

$$L^{-1}\left[s\bar{f}(s)\right] = \frac{d}{dt}\left[f(t)\right]$$

In general, $L^{-1}\left[s^n\bar{f}(s)\right] = \frac{d^n}{dt^n}\left[f(t)\right]$

Provided $f(0) = f'(0) = ... = f^{n-1}(0) = 0$

Example : $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$

Solution : $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$

$$= L^{-1}\left[s \times \frac{s}{(s^2+a^2)^2}\right]$$

Here $\bar{f}(s) = \dfrac{s}{(s^2 + a^2)^2}$

Using the above property, $L^{-1}[s\bar{f}(s)] = \dfrac{d}{dt}[f(t)]$, where f(0)=0

We have prove that, $L^{-1}\left[\dfrac{s}{(s^2 + a^2)^2}\right] = \dfrac{1}{2a} t \sin at = f(t)$ (say)

Since f(0) = 0

So by above property,

$$L^{-1}\left[s.\frac{s}{(s^2 + a^2)^2}\right] = \frac{d}{dt}\{f(t)\}$$

$$= \frac{d}{dt}\left\{\frac{1}{2a} t \sin al\right\}$$

$$= \frac{1}{2a}\{\sin at + t.\cos at.a\}$$

$$= \frac{1}{2a}\{\sin at + at \cos at\}$$

Property : IV

If $L^{-1}\{\bar{f}(s)\} = f(t), \text{then } L^{-1}\left\{\dfrac{\bar{f}(s)}{s}\right\} = \displaystyle\int_0^t f(t)dt$

Example : $L^{-1}\left[\dfrac{1}{(s^2 + a^2)^2}\right]$ result follows that $L^{-1}\left\{\dfrac{\bar{f}(s)}{s^2}\right\} = \displaystyle\int_0^t\int_0^t f(t)dt$

Solution : We have shown that

$$L^{-1}\left[\frac{s}{(s^2 + a^2)^2}\right] = \frac{1}{2a}(t \sin at) = f(t) \quad \text{(say)}$$

$$\therefore \quad L^{-1}\left[\frac{1}{(s^2 + a^2)^2}\right] = L^{-1}\left[\frac{1}{s} x \frac{s}{(s^2 + a^2)^2}\right]$$

$$= \int_0^t f(t)dt = \int_0^t \frac{t \sin at}{2a} dt$$

$$= \frac{1}{2a}\left[\int_0^t t \sin at \; dt\right]$$

$$= \frac{1}{2a}\left[t.\left(\frac{-\cos at}{a}\right) - \int_0^t 1\left(\frac{-\cos at}{a}\right)dt\right]$$

$$= \frac{1}{2a}\left[\frac{-t\cos at}{a} + \frac{\sin at}{a^2}\right]$$

$$= \frac{1}{2a^3}[\sin at - at \cos at]$$

Some Important Formulae :

- $\Gamma(n+1) = n\Gamma(n)$
- $\Gamma(1) = 1$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$
- $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

Laplace Transforms

1. $L(1) = \frac{1}{s}$
2. $L(t^n) = \frac{n!}{s^{n+1}} \quad n = 0,1,2,3$, ...
3. $L(e^{at}) = \frac{1}{s-a}$
4. L(sinat) = $\frac{a}{s^2+a^2}$
5. $L(\cos at) = \frac{s}{s^2+a^2}$
6. $L(\sin h\ at) = \frac{a}{s^2-a^2}$
7. $L(\cos h\ at) = \frac{s}{s^2-a^2}$

Inverse Laplace Transform

1. $L^{-1}\left(\frac{1}{s}\right) = 1$
2. $L^{-1}\left(\frac{1}{S^n}\right) = t^{n-1}, n = 1, 2, 3,....$
3. $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
4. $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$
5. $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
6. $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sin hat$
7. $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cos hat$

Application of first shifting property :

If L {f(t)} = $\bar{f}$ (s), then L {e_{at}f(t)}= $\bar{f}$ (s-a)

1. $L(e^{at}) = \frac{1}{s-a}$
2. $L(e^{at}t^n) = \frac{n!}{(s-a)^{n+1}}$
3. $L(e^{at}\sin bt) = \frac{b}{(s-a)^2+b^2}$
4. $L(e^{at}\cos bt) = \frac{s-a}{(s-a)^2+b^2}$
5. $L(e^{at}\sin hbt) = \frac{b}{(s-a)^2-b^2}$

1. $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
2. $L^{-1}\left[\frac{1}{(s-a)^n}\right] = \frac{e^{at}t^{n-1}}{(n-1)!}$
3. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b}e^{at}\sin bt$
4. $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cos bt$
5. $L^{-1}\left[\frac{1}{(s-a)^2-b^2}\right] = \frac{1}{b}e^{at}\sin h\ bt$

6. L (e_{at}cos h bt)= $\frac{s-a}{(s-a)^2-b^2}$

Where in each case s > a.

We use the following properties

6. $L^{-1}\left(\frac{s-a}{(s-a)^2-b^2}\right)=e^{at}\cos h\ bt$

Laplace Transform

1. If L{f(t)} = $\bar{f}$ (s), then

1a) $L\{t^n f(t)\}=(-1)^n \frac{d^n}{ds^n}\left[\bar{f}(s)\right]$

where n=1, 2, 3

1b) $L\{tf(t)\}=\frac{-d}{ds}\left(\bar{f}(s)\right)$

1c) Transform of integrals :

$L\left\{\int_0^t f(u)du\right\}=\frac{1}{s}\bar{f}(s)$

1d) $L\left\{\frac{f(t)}{t}\right\}=\int_s^\infty \bar{f}(s)ds$

Provided the integral exists

Inverse Laplace Transform

2. If $L^{-1}\{\bar{f}(s)\}=f(t)$, then

2a) $L^{-1}\{s^n\bar{f}(s)\}=\frac{d^n}{dt^n}\{f(t)\}$

Provided f(0)= $f'(0)=...=f^{n-1}(0)=0$

$L^{-1}\{s\bar{f}(s)\}=\frac{d}{dt}\{f(t)\}$

Provided f(0)=0

2b) $tf(t)=L^{-1}\left\{-\frac{d}{ds}\bar{f}(S)\right\}$

2c) $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\}=\int_0^t f(t)dt$

Also $L^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\}=\int_0^t\left\{\int_0^t f(t)dt\right\}dt$

$L^{-1}\left\{\frac{\bar{f}(s)}{s^3}\right\}=\int_0^t\left\{\int_0^t\left(\int_0^t f(t)dt\right)dt\right\}dt$

and so on

Important logn question with solution

Q1. Find $L^{-1}\left(\frac{5s+3}{(s-1)(s^2+2s+5)}\right)$

Ans. $L^{-1}\left(\frac{5s+3}{(s-1)(s^2+2s+5)}\right)$ can be solve by partial friction method

$$\left(\frac{5s+3}{(s-1)(s^2+2s+5)}\right)=\frac{A}{s-1}+\frac{Bs+c}{s^2+2s+5}$$

$$\Rightarrow \quad \frac{5s+3}{(s-1)(s^2+2s+5)}=\frac{A(s^2+2s+5)+(Bs+c)(s-1)}{(s-1)(s^2+2s+5)}$$

$\Rightarrow$ 5s+3 = A(s_2+2s+5)+(Bs+c)(S-1)

For s=1, 5, 1+3 = A(1_2 +2 x 1+5

$\Rightarrow$ 8 = 8A

$\Rightarrow$ A = 1

For s=0,

0x5+3 = A(0_2+2(0)+5)+(Bx0+C)(0-1)

$\Rightarrow$ 3 = 5A-C

$\Rightarrow$ C = 5A-3=5-3=2

$\Rightarrow$ C = 2

For s = -1,

-1x5+3=A {(-1)$_2$+2(-1)+5}+{B(-1)+c}(-1-1)

$\Rightarrow$ -2 = 4A+(-B+2)(-2)

$\Rightarrow$ -2 = 4A+2B-4

$\Rightarrow$ -2 = 4+2B-4 $\Rightarrow$ 2B=-2$\Rightarrow$B=-1

$$\therefore \quad \frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{2-s}{s^2+2s+5}$$

$$\Rightarrow L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\} = L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{2-s}{s^2+2s+5}\right\}$$

$$= L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{3-(s+1)}{(s+1)^2+2^2}\right\}$$

$$= e^t + L^{-1}\left\{\frac{3}{(s+1)^2+2^2}\right\} - L^{-1}\left\{\frac{(s+1)}{(s+1)^2+2^2}\right\}$$

$$= e^t + \frac{3}{2}e^{-t}\sin 2t - e^{-t}\cos 2t$$

Q2. Find $L^{-1}\left\{\frac{s+3}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s+3}{(s-2)^2+3^2}\right\}$

Ans. $L^{-1}\left\{\frac{s+3}{(s-2)^2+3^2}\right\} = L^{-1}\left\{\frac{s-2+5}{(s-2)^2+3^2}\right\}$

$$= L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2} + \frac{5}{(s-2)^2+3^2}\right\}$$

$$= L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + L^{-1}\left\{\frac{5}{(s-2)^2+3^2}\right\}$$

$$= e^{2t}\cos 3t + \frac{5}{3}e^{2t}\sin 3t$$

Q3. Find $L^{-1}\left[\tan^{-1}\frac{s}{2}\right]$

Ans. $L^{-1}\left[\tan^{-1}\frac{s}{2}\right] = \frac{-1}{t}L^{-1}\left\{\frac{d}{ds}\left(\tan^{-1}\frac{s}{2}\right)\right\}$

$$= \frac{-1}{t}L^{-1}\left\{\frac{1}{2}x\frac{1}{1+\left(\frac{s}{2}\right)^2}\right\}$$

$$= -\frac{1}{t}L^{-1}\left\{\frac{1}{2}x\frac{4}{4+s^2}\right\} = -\frac{1}{t}L^{-1}\left\{\frac{2}{s^2+4}\right\} = -\frac{1}{t}\sin 2t = \frac{-\sin 2t}{t}$$

Q4. Find $L^{-1}\left\{\frac{s}{(s-3)(s^2+4)}\right\}$

Ans. $L^{-1}\left\{\frac{s}{(s-3)(s^2+4)}\right\} = \frac{A}{s-3}+\frac{Bs+c}{s^2+4}$ ....(i)

$\Rightarrow \quad \frac{s}{(s-3)(s^2+4)} = \frac{A(s^2+4)+(Bs+c)(s-3)}{(s-3)(s^2+4)}$

$\Rightarrow \quad s = A(s_2+4) + (Bs+c)(s-3)$ .... (ii)

for s=3, equation(ii) becomes

$3 = A(3_2+4) \Rightarrow A = \frac{3}{13}$

For S = 0,

$0 = A(0_2+4)+(B.0+c)(0-3)$

$\Rightarrow \quad 0 = 4A-3c$

$\Rightarrow \quad 3C = 4A = 4x\frac{3}{13}$

$\Rightarrow \quad c = \frac{4}{13}$

For S =1,

$1 = A(1_2+4)+(B \text{ x } 1+c)(1-3)$

$\Rightarrow \quad 1 = 5A + (B+C)(-2)$

$\Rightarrow \quad 1 = 5x\frac{3}{13}+\left(B+\frac{4}{13}\right)(-2)$

$\Rightarrow \quad 1 = \frac{15}{13}-2B-\frac{8}{13} = \frac{7}{13}-2B$

$\Rightarrow \quad 2B = \frac{7}{13}-1 = \frac{-6}{13}$

$\Rightarrow \quad B = \frac{-3}{13}$

Putting A, B, C, in equation (i)

$\therefore \quad \frac{s}{(s-3)(s^2+4)} = \frac{3}{13(s-3)}+\frac{-3s+4}{13(s^2+4)}$

$\Rightarrow \quad L^{-1}\left\{\frac{s}{(s-3)(s^2+4)}\right\} = L^{-1}\left\{\frac{3}{13(s-3)}\right\}+L^{-1}\left\{\frac{-3s+4}{13(s^2+4)}\right\}$

$= \frac{3}{13}e^{3t}+L^{-1}\left\{\frac{-3s}{13(s^2+4)}\right\}+L^{-1}\left\{\frac{4}{13(s^2+4)}\right\}$

$= \frac{3}{13}e^{3t}-\frac{3}{13}\cos 2t+\frac{2}{13}\sin 2t$

Q5. Find $L^{-1}\left\{\log\left(\frac{1+s}{s}\right)\right\}$

Ans. We know $L\{t x f(t)\} = (-1)\frac{d\bar{f}(s)}{ds}$

$\Rightarrow \quad t\,f(t) = L^{-1}\left\{-\frac{d\bar{f}(s)}{ds}\right\}$

$\Rightarrow \quad f(t) = \frac{-1}{t}L^{-1}\left\{\frac{d\bar{f}(s)}{ds}\right\}$

$= -\frac{1}{t}\;L^{-1}\left\{\frac{1}{\frac{1+s}{s}}x\left(\frac{sx1-(1+s)}{s^2}\right)\right\}$

$= -\frac{1}{t}\;L^{-1}\left\{\frac{s}{s+1}x\frac{-1}{s^2}\right\}$

$= -\frac{1}{t}\;L^{-1}\left\{\frac{-1}{s(s+1)}\right\} = +\frac{1}{t}L^{-1}\left\{\frac{1}{s(s+1)}\right\}$

$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{1+s}$ ....(i)

$\Rightarrow \frac{1}{s(1+s)} = \frac{A(1+s)+Bs}{s(s+1)}$

$\Rightarrow$ 1=A(1+s) + BS .... (ii)

For s=0, 1=A(1+0)

$\Rightarrow$ A=1

For S=-1

1=B(-1) B=-1

$\therefore \quad \frac{1}{s(1+s)} = \frac{1}{s} - \frac{1}{1+s}$

$\Rightarrow \quad L^{-1}\left\{\log\left(\frac{1+s}{s}\right)\right\} = \frac{-1}{t}L^{-1}\left\{\frac{1}{s(1+s)}\right\}$

$= \frac{-1}{t}L^{-1}\left\{\frac{1}{s} - \frac{1}{1+s}\right\} = \frac{-1}{t}\{1-e^{-t}\}$

Q6. Find $L^{-1}\left\{\frac{s}{(s+3)^2+4}\right\} = L^{-1}\left\{\frac{(s+3)-3}{(s+3)^2+4}\right\}$

$= L^{-1}\left\{\frac{s+3}{(s+3)^2+4}\right\} - L^{-1}\left\{\frac{3}{(s+3)^2+4}\right\}$

$= e^{-3t}\cos 2t - \frac{3}{2}e^{-3t}\sin 2t$

Q1. Solve by method of laplace transforms $y''+4y'+3y=e^{-t}$, y(0)=(0)=1

Ans. Step - 1

Given differential equation is $y''+4y'+3y=e^{-t}$

Taking laplace transform of both sides, we get

$$L\{y''+4y'+3y\}=L\{e^{-t}\}$$

$$\Rightarrow \quad L\{y''\}+L\{4y'\}+L\{3y\}=L\{e^{-t}\}$$

$$\Rightarrow \quad \left(s^2\bar{y}(s)-sy(0)-y'(0)\right)+4\left(s\bar{y}(s)-y(0)\right)+3\bar{y}(s)=\frac{1}{s+1}$$

$$\Rightarrow \quad \bar{y}(s)(s^2+4s+3)-sy(0)-y'(0)-4y(0)=\frac{1}{s+1}$$

Step - 2 Putting y(0)=y'(0)=1

$$\Rightarrow \quad \bar{y}(s)(s^2+4s+3)-s-1-4=\frac{1}{s+1}$$

$$\Rightarrow \quad \bar{y}(s)(s^2+4s+3)-(s+5)=\frac{1}{s+1}$$

$$\Rightarrow \quad \bar{y}(s)(s^2+4s+3)=\frac{1}{s+1}+(s+5)$$

$$\Rightarrow \quad \bar{y}(s)=\frac{1+s^2+6s+5}{(s+1)(s^2+4s+3)}=\frac{s^2+6s+6}{(s+1)(s^2+4s+3)}$$

$$\Rightarrow \quad \bar{y}(s)=\frac{s^2+6s+6}{(s+1)(s+1)(s+3)}=\frac{s^2+6s+6}{(s+1)^2(s+3)}$$

Step - 3

Now solving R.H.S by partial fraction

$$\frac{s^2+6s+6}{(s+1)^2(s+3)}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{s+3} \quad \text{....................... (i)}$$

$$\Rightarrow \quad \frac{s^2+6s+6}{(s+1)^2(s+3)}=\frac{A(s+1)(s+3)+B(s+3)+c(s+1)^2}{(s+1)^2(s+3)}$$

$\Rightarrow$ s_2+6s+6 = A(s+1(s+3)+B(s+3)+C(s+1)$_2$ .... (ii)

For S= -1 equation(ii) becomes   for S=-3   For s = 0

1-6+6 = 2B   9-18+6=4C   6 = 3A+3B+C

$\Rightarrow$ 2B = 1 $\Rightarrow$ 4C = -3 $\Rightarrow$ $3A=6-\frac{3}{2}+\frac{3}{4}$ = $\frac{21}{4}$ $\Rightarrow$ $3A=\frac{21}{4}$

$\Rightarrow$ $B=\frac{1}{2}$ $\Rightarrow$ $C=\frac{-3}{4}$ $\Rightarrow$ $A=\frac{7}{4}$

Now putting A, B, C in Equation(i) becomes

$$\bar{y}(s)=\frac{s^2+6s+6}{(s+1)^2(s+3)}=\frac{7}{4(s+1)}+\frac{1}{2(s+1)^2}-\frac{3}{4(s+3)}$$

Step - 4

Taking inverse laplace transform on both side

$$y = L^{-1}\left\{\frac{7}{4(s+1)}\right\} + L^{-1}\left\{\frac{1}{2(s+1)^2}\right\} - L^{-1}\left\{\frac{3}{4(S+3)}\right\}$$

$y = \frac{7}{4}e^{-t} + \frac{1}{2}t.e^{-t} - \frac{3}{4}e^{-3t}$ Which is the desired result.

Q2. Solve by using laplace transformation method

$y''-3y+2y=e^{3t}$ **, y(0)=1,** $y'(0)=0$

Solution.

Step - 1 Given D.E is $y''-3y'+2y=e^{3t}$

Applying laplace transform we have

$$L(y'') - L(3y') + L(2y) = L\{e^{3t}\}$$

$$\Rightarrow \left(s^2\bar{y}(s) - sy(0) - y'(0)\right) - 3(s\bar{y}(s) - y(0)) + 2\bar{y}(s) = \frac{1}{s-3}$$

Step - 2

Now applying initial given condition y(0)=1, $y'(0)=0$

$$\Rightarrow \left(s^2\bar{y}(s) - s - 0\right) - 3(s\bar{y}(s) - 1) + 2\bar{y}(s) = \frac{1}{s-3}$$

$$\Rightarrow s^2\bar{y}(s) - s - 3s\bar{y}(s) + 3 + 2\bar{y}(s) = \frac{1}{s-3}$$

$$\Rightarrow \bar{y}(s)(s^2 - 3s + 2) = \frac{1}{s-3} + (s-3)$$

$$\Rightarrow \bar{y}(s)(s^2 - 3s + 2) = \frac{1+(s-3)^2}{s-3}$$

$$\Rightarrow \bar{y}(s) = \frac{1+(s-3)^2}{(s-3)(s^2-3s+2)}$$

$$= \frac{1+(s-3)^2}{(s-3)(s^2-2s-s+2)} = \frac{1+(s-3)^2}{(s-3)(s-2)(s-1)} = \frac{1+s^2+9-6s}{(s-3)(s-2)(s-1)}$$

$$\Rightarrow \bar{y}(s) = \frac{s^2-6s+10}{(s-3)(s-2)(s-1)}$$

Step - 3

Applying partial fraction on R.H.S

$$\frac{s^2-6s+10}{(s-3)(s-2)(s-1)}=\frac{A}{(s-3)}+\frac{B}{(s-2)}+\frac{C}{(s-1)} \quad \text{.......................... (i)}$$

$$\Rightarrow \quad \frac{s^2-6s+10}{(s-3)(s-2)(s-1)}=\frac{A(s-2)(s-1)+B(s-3)(s-1)+C(s-3)(s-2)}{(s-3)(s-2)(s-1)}$$

$\Rightarrow$ s_2 - 6s+10 = A(s-2)(s-1)+B(s-3)(s-1)+C(s-3)(s-2) .... (ii)

For s=2 in equation(ii) is

2_2-6x2+10=B(2-3)(2-1)

$\Rightarrow$ 2=-B

$\Rightarrow$ B=-2

For S=1

1-6+10=C(1-3)(1-2)

$\Rightarrow$ 5=2C

$\Rightarrow C=\frac{5}{2}$

For S=3

9-18+10=A(3-2)(3-1)

$\Rightarrow$ 1 = 2A

$\Rightarrow A=\frac{1}{2}$

Putting A, B, C In equation(i)

$$\bar{y}(s)=\frac{s^2+6s+10}{(s-3)(s-2)(s-1)}=\frac{1}{2(s-3)}+\frac{-2}{s-2}+\frac{5}{2(s-1)}$$

Step - 4

$$\bar{y}(s)=L(y)=\frac{1}{2(s-3)}+\frac{-2}{s-2}+\frac{5}{2(s-1)}$$

Applying laplace inverse both side

$$y=L^{-1}\left\{\frac{1}{2(s-3)}\right\}-2L^{-1}\left\{\frac{1}{s-2}\right\}+\frac{5}{2}L^{-1}\left\{\frac{1}{s-1}\right\}=\frac{1}{2}e^{3t}-2e^{2t}+\frac{5}{2}e^{t}$$

Q3. Solve by L. T method

$y''+2y'-3y=\sin t,$ **y(0)=** $y'(0)=0$

Ans.

Step - 1

Given D.E is

$y''+2y'-3y=\sin t,$

Applying laplace transform both side

$$\Rightarrow \quad L\{y''\}+2L\{y'\}-3L\{y\}=L\{\sin t\}$$

$$\Rightarrow \quad \left(s^2\bar{y}(s)-sy(0)-y'(0)\right)+2\left(s\bar{y}(s)-y(0)\right)-3\bar{y}(s)=\frac{1}{s^2+1}$$

$$\Rightarrow \quad \bar{y}(s)\left(s^2+2s-3\right)-sy(0)-y'(0)-2y(0)=\frac{1}{s^2+1}$$

Step - 2 Applying initial condition

$y(0)= y'(0)=0$

$\Rightarrow \quad \bar{y}(s)(s^2+2s-3)-0-0+0=\frac{1}{s^2+1}$

$\Rightarrow \quad \bar{y}(s)(s^2+2s-3)=\frac{1}{s^2+1}$

$\Rightarrow \quad \bar{y}(s)=\frac{1}{(s^2+1)(s^2+2s-3)}=\frac{1}{(s^2+1)(s^2+3s-s-3)}$

$\Rightarrow \quad \bar{y}(s)=\frac{1}{(s^2+1)(s+3)(s-1)}$

Step - 3

Now solving R.H.S by partial fraction

$$\frac{1}{(s^2+1)(s+3)(s-1)}=\frac{A}{(s+3)}+\frac{B}{(s-1)}+\frac{(Cs+D)}{s^2+1} \quad \text{.................... (i)}$$

$\Rightarrow \quad \frac{1}{(s^2+1)(s+3)(s-1)}=\frac{A(s-1)(s^2+1)+B(s+3)(s^2+1)+(cs+D)(s+3)(s-1)}{(s+3)(s-1)(s^2+1)}$

$\Rightarrow \quad 1 = A(s-1)(s_2+1)+B(s+3)(s_2+1)+(cs+D)(s+3)(s-1)$ .... (ii)

For s=1 Equation (ii) is

$1 = B.4.2$

$\Rightarrow \quad 8B=1$

$\Rightarrow \quad B=\frac{1}{8}$

For S=-3

$1=A(-4)(10)$

$\Rightarrow \quad -40A = 1$

$\Rightarrow \quad A=\frac{-1}{40}$

For S=0

$1=-A+3B-3D$

$\Rightarrow \quad 3D = \frac{1}{40}+\frac{3}{8}-1$

$\Rightarrow \quad 3D=\frac{-24}{40} \quad \Rightarrow D=\frac{-1}{5}$

For S=-1

$+1 = -4A+4B+(-C+D)(-4)$

$\Rightarrow \quad +1 = \frac{4}{40}+\frac{4}{8}+4c+\frac{4}{5}$

$\Rightarrow \quad 4c=1-\frac{1}{10}-\frac{1}{2}-\frac{4}{5}$

$\Rightarrow \quad 4c=\frac{10-1-5-8}{10}=\frac{10-14}{10}=\frac{-4}{10}=\frac{-2}{5}$

$\Rightarrow \quad 4c=\frac{-2}{5}$

$\Rightarrow \quad c=\frac{-2}{4x5}=\frac{-1}{10}$

Now equation(1) be comes

$$\frac{1}{(s+3)(s-1)(s^2+1)}=\frac{-1}{40(s+3)}+\frac{1}{8(s-1)}+\frac{\left(\frac{-1}{10}s-\frac{1}{s}\right)}{s^2+1}$$

$$\Rightarrow \quad L(y) = \frac{-1}{40(s+3)} + \frac{1}{8(s-1)} + \frac{\frac{-s-2}{10}}{(s^2+1)}$$

$$= \frac{-1}{40(S+3)} + \frac{1}{8(s-1)} - \frac{1}{10}\frac{s}{(s^2+1)} - \frac{1}{5(s^2+1)}$$

Step - 4 Applying iverse laplace

$$y = L^{-1}\left\{\frac{-1}{40(s+3)}\right\} + L^{-1}\left\{\frac{1}{8(s-1)}\right\} - \frac{1}{10}L^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{5}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\Rightarrow \quad y = \frac{-1}{40}e^{-3t} + \frac{1}{8}e^{t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t$$ which is the required result.

Q4. Solve

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t;\ x(0) = 2, \left.\frac{dx}{dt}\right]_{t=0} = -1$$

Ans. Taking laplace on both sides,

$$L\left\{\frac{d^2x}{dt^2}\right\} - 2L\left\{\frac{dx}{dt}\right\} + L\{x\} = L\{e^t\}$$

$$\Rightarrow \quad [s^2\bar{x}(s) - sx(0) - x'(0)] - 2[s\bar{x}(s) - x(0)] + \bar{x}(s) = \frac{1}{s-1}$$

$$\Rightarrow \quad s^2\bar{x}(s) - s.2 - (-1) - 2s\bar{x}(s) + 4 + \bar{x}(s) = \frac{1}{s-1}$$

$$\Rightarrow \quad \bar{x}(s)[s^2 - 2s + 1] - 2s + 5 = \frac{1}{s-1}$$

$$\Rightarrow \quad \bar{x}(s)[(s-1)^2] = \frac{1}{s-1} + 2s - 5$$

$$\Rightarrow \quad \bar{x}(s) = \frac{1}{(s-1)^3} + \frac{2s-5}{(s-1)^2} = \frac{1}{(s-1)^3} + \frac{2s-2-3}{(s-1)^2} = \frac{1}{(s-1)^3} + \frac{2(s-1)}{(s-1)^2} - \frac{3}{(s-1)^2}$$

$$\Rightarrow \quad \bar{x}(s) = \frac{1}{(s-1)^3} + \frac{2}{(s-1)} - \frac{3}{(s-1)^2}$$

$$\Rightarrow \quad L\{x(t)\} = \frac{1}{(s-1)^3} - \frac{3}{(s-1)^2} + \frac{2}{(s-1)}$$

$$\Rightarrow \quad x(t) = L^{-1}\left\{\frac{1}{(s-1)^3}\right\} - 3L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + 2L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= e^t\frac{t^2}{2!} - 3e^t.\frac{t}{1!} + 2e^t = t^2\frac{e^t}{2} - 3te^t + 2e^t$$

Q5. ; $\frac{d^2x}{dt^2} + 9x = \cos 2t$ $X(0)=1$ $x\left(\frac{\pi}{2}\right) = -1$

Ans. Let us choose $x'(0) = a$

taking laplace on both side in equation (i)

$$L\left\{\frac{d^2x}{dt^2}\right\} + 9L\{x\} = L\{\cos 2t\}$$

$\Rightarrow$ $$s^2\bar{x}(s) - sx(0) - x'(0) + 9\bar{x}(s) = \frac{s}{s^2+4}$$

$\Rightarrow$ $$s^2\bar{x}(s) - s.1 - a + 9\bar{x}(s) = \frac{s}{s^2+4}$$

$\Rightarrow$ $$\bar{x}(s)(s^2+9) - (s+a) = \frac{s}{s^2+4}$$

$\Rightarrow$ $$\bar{x}(s)(s^2+9) = \frac{s}{s^2+4} + s + a$$

$\Rightarrow$ $$\bar{x}(s) = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{a}{s^2+9}$$

$\Rightarrow$ $$L\{x(t)\} = \frac{1}{5}\left[\frac{5s}{(s^2+4)(s^2+9)}\right] + \frac{s}{s^2+9} + \frac{a}{s^2+9}$$

$\Rightarrow$ $$L\{x(t)\} = \frac{1}{5}\left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right] + \frac{s}{s^2+9} + \frac{a}{s^2+9}$$

$\Rightarrow$ $$x(t) = \frac{1}{5}L^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{1}{5}L^{-1}\left\{\frac{s}{s^2+9}\right\} + L^{-1}\left\{\frac{s}{s^2+9}\right\} + L^{-1}\left\{\frac{a}{s^2+9}\right\}$$

$$= \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{a}{3}\sin 3t$$

$\Rightarrow$ $$x(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{a}{3}\sin 3t$$

According to the question $x\left(\frac{\pi}{2}\right) = -1$

$\Rightarrow$ $$-1 = \frac{1}{5}\cos\left(2.\frac{\pi}{2}\right) + \frac{4}{5}\cos\left(3.\frac{\pi}{2}\right) + \frac{a}{3}\sin\left(\frac{3\pi}{2}\right)$$

$\Rightarrow$ $$-1 = -\frac{1}{5} + \frac{4}{5}.0 - \frac{a}{3} = -\left(\frac{1}{5} + \frac{a}{3}\right)$$

$\Rightarrow$ $$\frac{1}{5} + \frac{a}{3} = 1$$

$\Rightarrow$ $\frac{a}{3} = 1 - \frac{1}{5}$ $\Rightarrow$ $\frac{a}{3} = \frac{4}{5}$ $\Rightarrow$ $a = \frac{12}{5}$

Hence

$$x(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{12}{5}.\frac{1}{3}\sin 3t$$

$\Rightarrow$ $$x(t) = \frac{\cos 2t}{5} + \frac{4\cos 3t}{5} + \frac{4}{5}\sin 3t$$

Chapter - 5
Fourier Series

Some important Formulae -

1) $\sin n\pi = \sin(-n\pi) = 0$

2) $\cos n\pi = \cos(-n\pi) = (-1)^n$

3) $\sin(2n+1)\frac{\pi}{2} = (-1)^n$

4) $\cos(2n+1)\frac{\pi}{2} = 0$

5) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

6) $\int e^{ax} dx = \frac{e^{ax}}{a} + C$

7) $\int \sin nx\, dx = -\frac{\cos nx}{n} + C$

8) $\int \cos nx\, dx = \frac{\sin nx}{n} + C$

9) by parts formulae

$$\int (u.v)dx = u\int v.dx - \int \left[\frac{du}{dx}.\int v.dx\right]dx$$

Where u is 1st function and v is 2nd function followed by ILATE rule.

10) $\int e^{ax}.\sin bx.dx = \frac{e^{ax}}{a^2+b^2}$ [a sin bx - b cos bx]+C

11) $\int e^{ax}.\cos bx.dx = \frac{e^{ax}}{a^2+b^2}$ [a cos bx + b sin bx]+C

12) $\int_{\alpha}^{\alpha+2\pi} \cos nx.dx = \int_{\alpha}^{\alpha+2\pi} \sin nx.dx = 0$

13. $\int_{\alpha}^{\alpha+2\pi} \cos mx.\cos nx.dx = \int_{\alpha}^{\alpha+2\pi} \sin mx.\cos nx.dx$

$= \int_{\alpha}^{\alpha+2\pi} \sin mx.\sin nx.dx = \int_{\alpha}^{\alpha+2\pi} \sin nx.\cos mx.dx = 0$

14. $\int_{\alpha}^{\alpha+2\pi} \sin^2 nx.dx = \int_{\alpha}^{\alpha+2\pi} \cos^2 nx.dx = \pi$

Introduction

In many engineering problem, especially in the study of periodic phenomenon in SHM, conduction of heat, electrodynamics and acoustics, it is necessary to express a function in a series of sine and cosine. Most of the single valued functions which occur in applied mathmatics can be expressed in the form

$\frac{1}{2} a_0 + a_1 \cos x + a_2 \cos 2x +$ ....

$+ b_1 \sin x + b_2 \sin 2x +$ ....

within a desired range of values of the variable. Such a series is known as the fourier series.

5.1 Periodic functions :

A function f(x) defined for all real x is said to be periodic if there exists some positive number T such that f(x+T) = f(x) $\forall$ x, where T is the fundamental period of f(x).

The smallest positive value of T is the fundamental period of f(x)

For example : $\sin x = \sin(2\pi+x) = \sin(4\pi+x) =$ ....

$\cos x = \cos(2\pi+x) = \cos(4\pi+x) =$ ....

So sinx and cosx are periodic function of period 2π

$\tan x = \tan(\pi+x) =$ ....

$\cot x = \cot(\pi+x) =$ ....

So tanx and cot x are periodic functions of period π.

5.2 Fourier Series

If f(x) be a single valued periodic function having period 2π defined in the interval $\alpha < x < \alpha + 2\pi$, then the fourier series for f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where a_0, a_n and b_n are called fourier co-efficients given by Euler's formula

5.3 Dirichlet's condition for fourier expanssion of a function :

A function f(x) can be expressed as a fouries series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.\cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ where a_0, a_n and b_n are constant. if

a) f(x) is period, finite and single valued function
b) f(x) has a finite no. of discontinuities in any one period.
c) f(x) has at most a finite no. of maxima and minima.

5.4 Euler's Formulae :

The fourier series for the function f(x) is the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=}^{\infty} b_n \sin nx$$

Where
$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x).dx$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x).\cos nx.dx$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x).\sin nx.dx$$

These values of a_0, a_n, b_n are known as Eulers formulae.

If $\alpha = 0$, $a_o = \frac{1}{\pi}\int_0^{2\pi} f(x).dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\cos nx.dx$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\sin nx.dx$

If $\alpha = -\pi$ $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nx.dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\sin nx.dx$

5.5 Even and Odd function :

A) A function f(x) is said to be even if f(-x) = f(x)

For example : cos x, secx, x_2, x_4 .... are even functions.

B) A function is said to be odd if f(-x) = -f(x)

For example : sinx, cosecx, tanx, cotx, x, x_3, x_5 .... are odd function.

Note : 1. even function x even function = even function.

2. even function x odd function = odd function

3. odd function x odd function = even function.

4. $\int_{-a}^{a} f(x)dx = 0$ when f(x) is odd function

5. $\int_{-a}^{a} f(x)dx = 2\times\int_{0}^{a} f(x)dx$, when f(x) is an even function.

Example : Obtain the Fourier series for f(x) = e_{-x} 0<x<2π.

Ans. :

Step - 1 Write the function f(x) . Here f(x) = e_{-x}

Step - 2 Write the interval, here interval is (0, 2π)

Let $e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Step - 3 Find a_0, a_n and b_n using Euler's formula.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}dx = \frac{1}{\pi}\left[-e^{-x}\right]_0^{2\pi} = -\frac{1}{\pi}\left[e^{-2\pi} - e^0\right] = \frac{1-e^{-2\pi}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x`}\cos nx dx$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-x}(-\cos nx + n.\sin nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left\{\left[e^{-2\pi}(-\cos(-2n\pi) + n\sin(-2n\pi)\right] - e^0\left[-\cos 0 + n\sin 0\right]\right\}$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}(-1+n.0) - 1(-1+0)\right]$$

$$= \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{n^2+1}$$

$$a_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{2},\ a_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{5}\ ....$$

Finally,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin.nx.dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx dx$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-x}(-\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left\{\left[e^{-2\pi}(-\sin 2n\pi - n\cos 2n\pi)\right] - \left[e^{-0}(-\sin 0 - n\cos 0)\right]\right\}$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}(-0-n)-1(-n)\right]$$

$$= \frac{1}{\pi(n^2+1)}\left[n-ne^{-2\pi}\right] = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{n}{n^2+1}$$

$$\therefore \quad b_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{2},\ b_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{2}{5} \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Step - 4 Substituting the values of a_0, a_n, b_n in (1) we get

$$e^{-x} = \frac{a_0}{2} + \sum a_n.\cos nx + \sum b_n.\sin x$$

$$= \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{2} + \left[\left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{\cos x}{2} + \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{\cos 2x}{5} + \ldots\ldots\right]$$

$$+ \left[\left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{\sin x}{2} + \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{2}{5}\sin 2x + \ldots\ldots.\right]$$

Example : Find a Fourier series to represent x - x_2 from x = -π to x =π

Solution :

Let $$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)dx = \frac{1}{\pi}\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right) - \left(\frac{\pi^2}{2} + \frac{\pi^3}{3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3}\right] = -\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx.dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - \int(1-2x)\frac{\sin nx}{n}dx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - \frac{1}{n}\left[(1-2x)\left(-\frac{\cos nx}{n}\right) - \int -2\left(-\frac{\cos nx}{n}\right)dx\right]\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - (1-2x)x\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[(\pi-\pi^2)\frac{\sin n\pi}{n} + (1-2\pi)\frac{\cos n\pi}{n^2} + \frac{2}{n^3}\sin n\pi\right] - \left[(-\pi-\pi^2)\frac{\sin(-n\pi)}{n} + (1+2\pi)\frac{\cos(-n\pi)}{n^2} + \frac{2}{n^3}\sin(-n\pi)\right]\right\}$$

$$= \frac{1}{\pi}\left\{\left[(\pi-\pi^2).0 + (1-2\pi)\frac{(-1)^n}{n^2} + \frac{2}{n^3}.0\right] - \left[-(\pi+\pi^2).0 + (1+2\pi)\frac{(-1)^n}{n^2} + \frac{2}{n^3}.0\right]\right\}$$

[cos (-nπ)= cos n π = $(-1)_n$]
sin(-nπ)= - sin nπ = 0

$$= \frac{1}{\pi}\left\{\left[0 + (1-2\pi)\frac{(-1)^n}{n^2} + 0\right] - \left[0 + (1+2\pi)\frac{(-1)^n}{n^2}\right]\right\}$$

$$= \frac{1}{\pi}\left[(1-2\pi)\frac{(-1)^n}{n^2} - (1+2\pi)\frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - 2\pi\frac{(-1)^n}{n^2} - \frac{(-1)^n}{n^2} - 2\pi\frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left(-4\pi\frac{(-1)^n}{n^2}\right) = \frac{-4(-1)^n}{n^2}$$

$$a_1 = \frac{4}{1^2}, \quad a_2 = \frac{-4}{2^2} \qquad a_3 = \frac{4}{9^2}..................$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(-\frac{\cos nx}{n}\right) - \int(1-2x)\left(-\frac{\cos nx}{n}\right)dx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(-\frac{\cos nx}{n}\right) + \frac{1}{n}\left\{(1-2x)\frac{\sin nx}{n} - \int -2\frac{\sin nx}{n}dx\right\}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(-\frac{\cos nx}{n}\right) - (1-2x)x\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[(\pi - \pi^2)\left(-\frac{\cos n\pi}{n}\right) + (1-2\pi)\left(\frac{\sin n\pi}{n^2}\right) - \frac{2}{n^3}\cos n\pi\right]\right.$$

$$\left. - \left[-(\pi + \pi^2)\left(-\frac{\cos(-n\pi)}{n}\right) + (1+2\pi)\frac{\sin(-n\pi)}{n^2} - 2\frac{\cos(-n\pi)}{n^3}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\left[-(\pi - \pi^2)\frac{(-1)^n}{n} + (1-2\pi).0 - \frac{2}{n^3}(-1)^n\right] - \left[(\pi + \pi^2)\frac{(-1)^n}{n} + (1+2\pi).0 - 2\frac{(-1)^n}{n^3}\right]\right\}$$

$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n} + \pi^2\frac{(-1)^n}{n} - \frac{2}{n^3}(-1)^n - \pi\frac{(-1)^n}{n} - \pi^2\frac{(-1)^n}{n} + 2\frac{(-1)^n}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n} - \pi\frac{(-1)^n}{n}\right] = \frac{1}{\pi}\left[-2\pi\frac{(-1)^n}{n}\right]$$

$$= -2\frac{(-1)^n}{n}$$

$$b_1 = \frac{2}{1}, \; b_2 = \frac{-2}{2}, \; b_3 = \frac{2}{3}, \; b_4 = \frac{-2}{4}$$ and so on

Substituting the values of a_n's and b_n's in (i), we get

$$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + ...\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + ...\right]$$

Fourier expansion of even and odd functions

We have already discussed even and odd function

We know a periodic function f(x) defined in $(-\pi,\pi)$ can be represented by fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\;dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\;dx = 0$$

Case - I When f(x) is even function

f(x) is an even function and sin nx is odd function, so f(x). sin nx is odd function

$\therefore \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\sin nx \, dx = 0$

Its Fourier expansion contains only a_0 and a_n.

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx$$

Case II when f(x) is an odd function,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx = 0$$

(since f(x) is odd function and cos nx is even function, so f(x) cos nx is odd function)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx$$

Thus if a periodic function f(x) is odd the fourior expansion contains only b_n

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx$

$\therefore$ Its fourier expansion contains only b_n

Example : Find the fourier seris to represent the functin f(x) = |cos x|, $-\pi \le x \le \pi$

Ans. If f(x) = |cosx|, expand f(x) as a fourier series in the interval $(-\pi, \pi)$.

As f (-x) = |cos(-x)|=|cosx| = f(x),

So, |cos x| is an even function.

$\therefore \quad f(x) = \frac{a_0}{2} + \sum a_n \cos nx$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} |\cos x| dx = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} (-\cos x) dx\right]$ [cos x is -ve when $\pi/2 < x < \pi$]

$$= \frac{2}{\pi}\left\{|\sin x|_0^{\pi/2} - |\sin x|_{\pi/2}^{\pi}\right\} = \frac{2}{\pi}[(1-0)-(0-1)] = \frac{4}{\pi}$$

and $a_n = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx \, dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos nx \, dx + \int_{\pi/2}^{\pi} (-\cos x)\cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} [\cos(n+1)x + \cos(n-1)x] \, dx - \int_{\pi/2}^{\pi} [\cos(n+1)x + \cos(n-1)x] \, dx\right]$$

$$= \frac{1}{\pi}\left\{\left|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_0^{\pi/2} - \left|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right\} + \left\{\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right\}\right]$$

$$= \frac{2}{\pi}\left(\frac{\cos n\pi/2}{n+1} - \frac{\cos n\pi/2}{n-1}\right) = \frac{-4\cos n\pi/2}{\pi(n^2-1)} (n \ne 1)$$

In particular
$$a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x dx - \int_{\pi/2}^{\pi} \cos^2 x\, dx\right] = 0$$

Hence
$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left\{\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + ...\right\}$$

5.6 Fourier series for function having points of discontinuity :

If the function f(x) have finite no. of discontinuities in one period i.e. its graph may consist of a finite no. of different curves given by different equations, then also it can be expressed in fourier series.

Mathematically : If in the interval $(\alpha, \alpha + 2\pi)$, f(x) is defined by

$$f(x) = \begin{cases} \phi(x), \alpha < x < c \\ \psi(x), c < x < \alpha + 2\pi \end{cases}$$

i.e c is the point of discontinuity

then
$$a_0 = \frac{1}{\pi}\left[\int_\alpha^c \phi(x)dx + \int_c^{\alpha+2\pi} \varphi(x)dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_\alpha^c \phi(x)\cos nx dx + \int_c^{\alpha+2\pi} \varphi(x)\cos nx dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_\alpha^c \phi(x)\sin nx dx + \int_c^{\alpha+2\pi} \varphi(x)\sin nx dx\right]$$

Example :

Find the Fourier series expansion for f(x), if

$$f(x) = \begin{cases} -\pi, -\pi < x < 0 \\ x, 0 < x < \pi \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$

Solution :

Let f(x)= $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then
$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^0 (-\pi)dx + \int_0^\pi x dx\right] = \frac{1}{\pi}\left[-\pi[x]_{-\pi}^0 + [x^2/2]_0^\pi\right] = \frac{1}{\pi}\left(-\pi^2 + \pi^2/2\right) = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^0 (-\pi)\cos nx dx + \int_0^\pi x\cos nx dx\right] = \frac{1}{\pi}\left[-\pi\left[\frac{\sin nx}{n}\right]_{-\pi}^0 + \left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^\pi\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1)$$

$\therefore$
$$a_1 = \frac{-2}{\pi.1^2}, a_2 = 0, a_3 = -\frac{2}{\pi.3^2}, a_4 = 0, a_5 = -\frac{2}{\pi.5^2} \text{ etc.}$$

Finally
$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^0 (-\pi)\sin nx dx + \int_0^\pi x\sin nx dx\right] = \frac{1}{\pi}\left[\pi\left[\frac{\cos nx}{n}\right]_{-\pi}^0 + \left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^\pi\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1 - 2\cos n\pi)$$

$\therefore$ $b_1 = 3$, $b_2 = -\frac{1}{2}$, $b_3 = 1$, $b_4 = -\frac{1}{4}$, etc.

Hence substituting the values of a_n's and b_n's in (i), we get

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \quad ....(ii)$$

which is the required result.

Putting x = 0 in (ii), we obtain $f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + ...\infty\right)$ ...(iii)

Now f(x) is discontinuous at x = 0. As a matter of fact $\left[\quad f(x) = \frac{1}{2}\{f(x+h) + f(x-h)\}\right]$

f(0 - 0) = $-\pi$ and f(0+0)=0 $\quad \therefore f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = \frac{-\pi}{2}$

Hence (iii) takes the form $-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right]$

$$\Rightarrow \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$$

(Proved)

Long Questions with Answer

1. Express f(x) = x as a Fourier Series in The interval $-\pi \le x \le \pi$

Solution : The given function is

f(x) = x, $-\pi \le x \le \pi$

The fourier series of the given function is $x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\, dx$

$= \frac{1}{\pi}.\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right] = \frac{1}{\pi}.0 = 0$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\; dx = 0$

(Since f(x)is odd function and cosnx is even function, so f(x) cosnx is odd function)

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\, dx$

$= \frac{1}{\pi}\left[x.\int \sin nx\; dx - \int \frac{d}{dx}(x)\int \sin nx\, dx\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[x.\left(\frac{-\cos nx}{n}\right) + \int \frac{\cos nx}{n}dx\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n} + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi}$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2} - \frac{\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2\pi\cos n\pi}{n} + \frac{2\sin n\pi}{n^2}\right] \qquad \left[\begin{array}{l}\sin n\pi = 0 \\ \cos n\pi = (-1)^n\end{array}\right]$$

$$= \frac{2}{\pi}.\left(\frac{-\pi}{n}\right).(-1)^n$$

$$= \frac{2}{n}(-1)^{n+1}$$

So $f(x) = \frac{0}{2} + \sum_{n=1}^{\infty}.0\cos nx + \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx$

$$\Rightarrow x = 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + ...\right]$$

2. Obtain fourier series for f(x)=|x| in the interval $-\pi \le x \le \pi$.

Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$

Ans. Here f(x)=|x|

f(-x)=|-x|=|x|=f(x)

Hence f(x) is an even function

Fouries series of f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

We know $\int_{-a}^{a} f(x).dx = \begin{cases} 2\int_0^a f(x).dx, & f(x) \text{ is even} \\ 0, & f(x) \text{ is odd}\end{cases}$

$\therefore \quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

$$= \frac{2}{\pi}\int_0^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}.\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}.\left[\frac{x^2}{2} - 0\right] = \pi$$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} |x|\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[x.\frac{\sin nx}{n} - \int_0^{\pi}\frac{\sin nx}{n}dx\right]$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\left(\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

Hence f(x)= $\frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos.nx$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{-2}{1^2}\cos x - \frac{2}{3^2}\cos 3x - \frac{2}{5^2}\cos 5x........\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right]$$

Putting x = 0

$$f(0) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + \ldots\ldots..\right]$$

$$\Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots..\right]$$

$$\Rightarrow \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots..\right] = \frac{\pi}{2}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots.. = \frac{\pi}{2} x \frac{\pi}{4}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots.. = \frac{\pi^2}{8}$$

3. Obtain a fourier Series e_{-ax} from $x = -\pi$ to $x = \pi$.

Hence derive series for $\frac{\pi}{\sinh \pi}$

Solution : Given f(x) = e_{-ax}

Hence the fourier series for f(x) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{-1}{a\pi}\left[e^{-a\pi} - e^{a\pi}\right]$$

$$= \frac{1}{a\pi}\left[e^{a\pi} - e^{-a\pi}\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} a(x).\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx\, dx$$

$$\left[\quad \int e^{ax}\cos bx\ dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-a\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{-a\pi}(-a\cos n\pi + n\sin n\pi) - e^{a\pi}\{-a\cos(-n\pi) + n\sin(-n\pi)\}\right]$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{-a\pi}\{-a.(-1)^n + 0\} - e^{a\pi}\{-a(-1)^n - 0\}\right]$$

$$= \frac{1}{\pi(a^2+n^2)}.\left[-e^{-a\pi}.a(-1)^n + ae^{a\pi}.(-1)^n\right]$$

$$= \frac{a(-1)^n}{\pi(a^2+n^2)}.\left(e^{a\pi} - e^{-a\pi}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx\,dx$

$= \left[\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx \right]$

$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-a\sin nx - n\cos nx) \right]_{-\pi}^{\pi}$

$= \frac{1}{\pi(a^2+n^2)}\left[e^{-a\pi}(-a\sin n\pi - n\cos n\pi) - e^{a\pi}(-a\sin(-n\pi) - n\cos(-n\pi)) \right]$

$= \frac{1}{\pi(a^2+n^2)}\left[e^{-a\pi}\left(0 - n(-1)^n\right) - e^{a\pi}\left(0 - n(-1)^n\right) \right]$

$= \frac{1}{\pi(a^2+n^2)}\left[n(-1)^n e^{a\pi} - n(-1)^n e^{-a\pi} \right]$

$= \frac{n(-1)^n}{\pi(a^2+n^2)}\left[e^{a\pi} - e^{-a\pi} \right]$

Hence f(x) = $\frac{e^{a\pi} - e^{-a\pi}}{2a\pi} + \sum_{n=1}^{\infty} \frac{a(-1)^n}{\pi(a^2+n^2)}.(e^{a\pi} - e^{-a\pi}).\cos nx + \sum_{n=1}^{\infty} \frac{n(-1)^n}{\pi(a^2+n^2)}\left(e^{a\pi} - e^{-a\pi}\right)\sin nx$

$= \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty} \frac{2a(-1)^n}{\pi(a^2+n^2)}\sinh a\pi\cos nx + \sum_{n=1}^{\infty} \frac{2n(-1)^n}{\pi(a^2+n^2)}\sinh a\pi\sin nx$

$= \frac{2\sinh a\pi}{2a\pi} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi(a^2+n^2)}(a\sinh a\pi\cos nx) + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi(a^2+n^2)}\sinh a\pi\sin nx$

$= \frac{2\sinh a\pi}{\pi}\left\{ \frac{1}{2a} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n}{a^2+n^2}a\cos nx + \frac{(-1)^n}{a^2+n^2}n\sin nx \right] \right\}$

$= \frac{2\sinh a\pi}{\pi}\left[\left\{ \frac{1}{2a} - \frac{a\cos x}{1+a^2} + \frac{a\cos 2x}{4+a^2} - \frac{a\cos 3x}{9+a^2} + ... \right\} + \left\{ \frac{-\sin x}{1+a^2} + \frac{2\sin 2x}{4+a^2} - \frac{3\sin 3x}{9+a^2} + ... \right\} \right]$

4. Find the fourier series of the function

$$f(x) = \begin{cases} x^2, & 0 \le x \le \pi \\ -x^2, & -\pi \le x \le 0 \end{cases} = \begin{cases} -x^2, & -\pi \le x \le 0 \\ x^2, & 0 \le x \le \pi \end{cases}$$

Solution : The fourier series of given function is f(x)= $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx \right]$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x^2dx + \int_{0}^{\pi} x^2dx \right]$

$= \frac{1}{\pi}\left[\left(\frac{-x^3}{3}\right)_{-\pi}^{0} + \left(\frac{x^3}{3}\right)_{0}^{\pi} \right]$

$= \frac{1}{\pi}\left[-\frac{\pi^3}{3} + \frac{\pi^3}{3} \right]$ =0

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nx\ dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x^2\cos nx\,dx + \int_{0}^{\pi} x^2\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[-x^2\int\cos nx\,dx - \int\frac{d}{dx}(-x^2)\left(\int\cos nx\,dx\right)dx\right]_{-\pi}^{0} + \left[x^2\int\cos nx dx - \int\frac{d}{dx}(x^2).\left(\int\cos nx\,dx\right)dx\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{-x^2\sin nx}{n} + \int\frac{2x\sin nx}{n}dx\right]_{-\pi}^{0} + \left[\frac{x^2\sin nx}{n} - \int\frac{2x\sin nx\,dx}{n}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{-x^2\sin nx}{n} + \frac{2}{n}\left(\int x\sin nx dx\right)\right]_{-\pi}^{0} + \left[\frac{x^2\sin nx}{n} - \frac{2}{n}\left(\int x\sin nx dx\right)\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-x^2\sin nx}{n} + \frac{2}{n}\left(x\int\sin nx dx - \int\frac{d}{dx}(x).\left(\int\sin nx\,dx\right)dx\right)\right]_{-\pi}^{0}\right.$$

$$\left. + \left[\frac{x^2\sin nx}{n} - \frac{2}{n}\left(x\int\sin nx dx - \int\frac{d}{dx}(x)\left(\int\sin nx\,dx\right)dx\right)\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-x^2\sin nx}{n} + \frac{2}{n}\left(\frac{-x\cos nx}{n} - \int -\frac{\cos nx}{n}dx\right)\right]_{-\pi}^{0}\right.$$

$$\left. + \left[\frac{x^2\sin nx}{n} - \frac{2}{n}\left(\frac{-x\cos nx}{n} - \int\frac{-\cos nx}{n}dx\right)\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-x^2\sin nx}{n} - \frac{2}{n^2}x\cos nx + \frac{2}{n^3}\sin nx\right]_{-\pi}^{0}\right.$$

$$\left. + \left[\frac{x^2\sin nx}{n} + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin nx\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left\{\frac{-2}{n^2}\left(-(-\pi)\cos n\pi\right)\right\} + \frac{2}{n^2}\left\{\pi\cos n\pi - 0\right\}\right]$$

$(\quad \sin n\pi = 0$
$\sin n0 = 0$
$\& \cos n\pi = (-1)^n)$

$$= \frac{1}{\pi}\left[\frac{-2}{n^2}\pi(-1)^n + \frac{2}{n^2}.\pi.(-1)^n\right]$$

$$= \frac{1}{\pi}.\pi.\frac{2}{n^2}\left\{(-1)^n - (-1)^n\right\} = \frac{2}{n^2}\left\{(-1)^n - (-1)\right\} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x^2\sin nx\,dx + \int_{0}^{\pi} x^2\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{-x^2\int \sin nx\,dx - \int\left\{\frac{d}{dx}(-x^2)\right\}\left(\int \sin nx\,dx\right)\right\}_{-\pi}^{0}\right.$$

$$\left. + \left\{x^2\int \sin nx\,dx - \int\left\{\frac{d}{dx}x^2\right\}\left(\int \sin nx\,dx\right)dx\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{x^2\cos nx}{n} - \int\frac{2x}{n}\cos nx\,dx\right\}_{-\pi}^{0} + \left\{-\frac{x^2\cos nx}{n} + \int\frac{2x}{n}\cos nx\,dx\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{x^2\cos nx}{n} - \frac{2}{n}\left(x\int \cos nx dx - \int\left(\frac{d}{dx}x\right)\left(\int \cos nx dx\right)dx\right)\right\}_{-\pi}^{0}\right.$$

$$\left. + \left\{-\frac{x^2\cos nx}{n} + \frac{2}{n}\left(x\int \cos nx dx - \int\left(\frac{d}{dx}x\right)\left(\int \cos nx dx\right)dx\right)\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{x^2\cos nx}{n} - \frac{2x\sin nx}{n^2} + \frac{2}{n^2}\int \sin nx\,dx\right\}_{-\pi}^{0}\right.$$

$$\left. + \left\{\frac{-x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} - \frac{2}{n^2}\int \sin nx\,dx\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{x^2\cos nx}{n} - \frac{2x\sin nx}{n^2} - \frac{2\cos nx}{n^3}\right\}_{-\pi}^{0} + \left\{-\frac{x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(0 - 0 - \frac{2\cos 0}{n^3}\right) - \left(\frac{\pi^2\cos n\pi}{n} - \frac{2\pi\sin n\pi}{n^2} - \frac{2\cos n\pi}{n^3}\right)\right.$$

$$\left. + \left(-\frac{\pi^2\cos n\pi}{n} + \frac{2\pi\sin n\pi}{n^2} + \frac{2\cos n\pi}{n^3}\right) - \left(0 + 0 + \frac{2\cos 0}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-2}{n^3} - \frac{\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{4}{n^3} + \frac{4(-1)^n}{n^3} - \frac{2\pi^2(-1)^n}{n}\right] \qquad = \frac{2}{\pi}\left[\frac{2}{n^3}\{(-1)^n - 1\} - \frac{\pi^2(-1)^n}{n}\right]$$

$$\therefore\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{2}{n^3}\{(-1)^n - 1\} - \frac{\pi^2(-1)^n}{n}\right]\sin nx$$

Q.5 Find the fourier series to represent the function $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x < \pi \end{cases}$

Solution : The fourier of given function is f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x^2\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \left[\frac{x^3}{3}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} - \frac{0^3}{3}\right] = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cos nx\,dx + \int_{0}^{\pi} x^2\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\int \cos nx\,dx - \int\left(\frac{d}{dx}x^2\right)\left(\int \cos nx\,dx\right)dx\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{x^2\sin nx}{n} - \int \frac{2x\sin nx}{n}\,dx\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{x^2\sin nx}{n} - \frac{2}{n}\left\{\int x\sin nx\,dx\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{x^2\sin nx}{n} - \frac{2}{n}\left\{x\int \sin nx\,dx - \int\left(\frac{d}{dx}x\right)\left(\int \sin nx\,dx\right)dx\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{x^2\sin nx}{n} - \frac{2}{n}\left\{-\frac{x\cos nx}{n} - \int -\frac{\cos nx}{n}\,dx\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin nx\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\left\{\frac{\pi^2\sin n\pi}{n} + \frac{2\pi\cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right\} - \left\{\frac{0\sin n0}{n} + \frac{2}{n^2}0\cos n0 - \frac{2}{n^3}\sin no\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2.0}{n} + \frac{2\pi(-1)^n}{n^2} - \frac{2.0}{n^3} - (0+0-0)\right] = \frac{1}{\pi}.\frac{2\pi(-1)^n}{n^2}$$

$$\Rightarrow \quad a_n = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\sin nx\,dx + \int_{0}^{\pi} x^2\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x^2\sin nx\,dx = \frac{1}{\pi}\left[x^2\int \sin nx\,dx - \int\left(\frac{d}{dx}x^2\right)\left(\int \sin nx\,dx\right)dx\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x^2\cos nx}{n} - \int 2x\left(\frac{-\cos nx}{n}\right)dx\right]_0^\pi$$

$$= \frac{1}{\pi}\left[-\frac{x^2\cos nx}{n} + \frac{2}{n}\int x\cos nx\,dx\right]_0^\pi$$

$$= \frac{1}{\pi}\left[-\frac{x^2\cos nx}{n} + \frac{2}{n}\left\{x\int \cos nx dx - \int\left(\frac{d}{dx}x\right)\left(\int \cos nx\,dx\right)dx\right\}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{-x^2\cos nx}{n} + \frac{2}{n}\left\{\frac{x\sin nx}{n} - \int\frac{\sin nx}{n}dx\right\}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{-x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\left\{\frac{-\pi^2\cos n\pi}{n} + \frac{2\pi\sin n\pi}{n^2} + \frac{2\cos n\pi}{n^3}\right\} - \left\{\frac{-0^2\cos n0}{n} - \frac{2.0\sin n0}{n^2} + \frac{2\cos n0}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + 0 + \frac{2(-1)^n}{n^3} + 0 + 0 - \frac{2}{n^3}\right]$$

$$\Rightarrow\ b_n = \frac{1}{\pi}\left[\frac{2(-1)^n}{n^3} - \frac{2}{n^3} - \frac{\pi^2(-1)^n}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{n^3}\left\{(-1)^n - 1\right\} - \frac{\pi^2(-1)^n}{n}\right]$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$= \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2.(-1)^n\cos nx}{n^2} + \sum_{n=1}^{\infty}\frac{1}{\pi}\left[\frac{2}{n^3}\left\{(-1)^n - 1\right\} - \frac{\pi^2(-1)^n}{n}\right]\sin nx$$

$$= \frac{\pi^2}{6} + \left[\frac{-2\cos x}{1^2} + \frac{2\cos 2x}{2^2} - \frac{2\cos 3x}{3^2} + ...\right] + \frac{1}{\pi}\left[\left(\frac{2}{1^3}(-2) + \pi^2\right)\sin x + \left(\frac{-\pi^2}{2}\right)\sin 2x\right.$$
$$\left. + \left(\frac{2}{3^3}(-2) + \frac{\pi^2}{3}\right)\sin 3x + ...\right]$$

$$= \frac{\pi^2}{6} - 2\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2}.....\right] - \frac{1}{\pi}\left[\left(\frac{-4}{1^3} + \pi^2\right)\sin x\right.$$
$$\left. + \left(\frac{-\pi^2}{2}\right)\sin 2x + \left(\frac{-4}{3^3} + \frac{\pi^3}{3}\right)\sin 3x + ...\right]$$

Chapter - 6
Numerical Methods

6.1 Algebraic and transcedental equation

The equation f(x) = 0 is algebraic if f(x) is purely a polynomial in x.

For example x_3 - x +1 = 0

A nonalgebraic equation is called a transcedental equation.

Example cosx - xe_x = 0

Introduction :

In scientific and engineering studies, a frequently occuring problem is to find the roots of equation of the form f(x) = 0. If f(x) is a quadratic, cubic or biquadratic expression, then algebraic formulae are available for finding the roots; on the otherhand if f(x) is a polynomial of higher degree or an expression involving transcendental function, algebraic methods are not availble for finding roots in such cases we have to take help of numerical methods for finding an approximate root of the equation.

6.2 Iterative Method :

This method consists of repeated execution of the same process, where at each step the result of the preceeding step is used. This is known as iteration process and is repeated till the result is obtained to a desired degree of accuracy.

For finding the solution by Numerical Method.

There are different types of methods but in our syllabus, we have

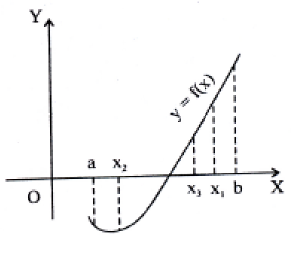
i) Bisection method

ii) Newton-Raphson method (N-R mehtod)

6.2.1 Bisection method :

If a function f(x) is continuous in the closed interval [a,b], f(a) and f(b) are of opposite signs, Then there exists at least one real root of f(x) = 0, between a and b.

Fig.



Working Procedure :

Let f(x) = 0 be either algebraic or transcendental equation. Then successive approximation to its roots are obtained as per the following steps.

Step - 1 Choose two values x = a and x = b such taht f(a) and f(b) are of oposite signs i.e.

f(a) x f(b) < 0

Step - 2 If a<b, then the root lies in the interval [a,b] and the 1st approximate root is the midpoint of [a,b] i.e. $x_1 = \frac{a+b}{2}$

Step - 3 If $f(x_1) = 0$, then x_1 is the root of the equation;

else if f(a) and $f(x_1)$ are of opposite signs, then 2nd approximate root is $x_2 = \frac{a+x_1}{2}$.

else if $f(x_1)$ and f(b) are of opposite signs then the 2nd approximate root is $x_2 = \frac{b+x_1}{2}$

Step - 4 Repeat the above procedure to obtain the successive approximate roots x_3, x_4 and so on. Until getting desired accuracy. i.e. correct upto one decimal places or two decimal places or in three stages or four stages likewise.

Note : Correct upto one decimal point means the 1st digit after decimal point is same in previous stage and current stage. Correct upto 2 decimal places means the 1st two digits after decimal point are same in previous stage and current stage. Three or four stages means repeat the method three times or four times.

Note : We have to choose the initial interval where the root lies by trial method we can test both for -ve and +ve values of x. It is convenient to calculate for small no. i.e. -1, 0, 1, 2 ....

Example :

Find real root of equation $x_3 - 4x - 9 = 0$ using the bisection method in four stages.

Solution :

Step 1 : Given $f(x) = x_3 - 4x - 9$

$f(0) = -9 < 0$

$f(1) = 1-4-9 = -12 < 0$

$f(2) = -9 < 0$

$f(3) = 6 > 0$

$f(2) \times f(3) < 0$ so root lies between 2 & 3

Step 2 : $x_1 = \frac{2+3}{2} = 2.5$

$f(2.5) = -3.375 < 0$

$f(2.5) \times f(3) < 0$ so root lies between 2.5 & 3

Step 3 : $x_2 = \frac{2.5+3}{2} = 2.75$

$f(2.75) = 0.796 > 0$

$f(2.75) \times f(2.5) < 0$ so root lies between 2.5 and 2.75

Step 4 : $x_3 = \frac{2.75+2.5}{2} = 2.625$

$f(2.625) = -1.4120 < 0$

$f(2.625) \times f(2.75) < 0$ so root lies between 2.625 and 2.75

Step 5 : $x_4 = \frac{2.625+2.75}{2} = 2.6875$

$f(2.6875) = -0.3391 < 0$

hence, the real root is 2.6 correct upto two significant figure in four stages.

6.2.2 Newton - Raphson Method :

Derivation of Newton's formula for finding the root of f(x)= 0

Let x_0 be an approximate root of the equation f(x) = 0. If $x_1 = x_0 + h$ be the exact root, then $f(x_1)= 0$.

By expanding $f(x_0+h)$ by Taylors series.

$$f(x_0+h) = f(x_0)+h f'(x_0) + \frac{h^2}{2!} f''(x_0) + ... = 0$$

Since h is very small, neglecting h_2 and higher power of h, we get

$$f(x_0) + hf'(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

A closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

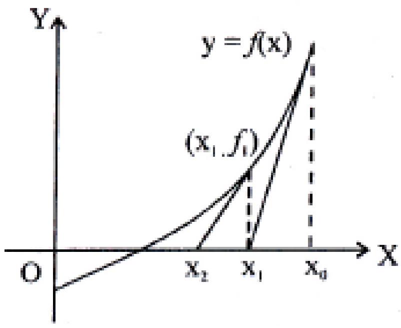
.... in general $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Which is known as Newton Raphson formula or Newton's Iteration formula.

Note : The order of convergence of NR method is 2

Uses : This method is convenient to use when the curve y = f(x) intersects x-axis is nearly vertical.

Fig



Working procedure for NR method

Let f(x)=0 be the algebraic or transcendental equation.

Then the successive approximations to the root of the equation are obtained as per the following steps.

Step 1 : Find $f'(x)$

Step 2 : Choose x = a and x= b be two values of x such that f(a) and f(b)are of opposite sign.

Step 3 : Choose x_0 [a, b] as the initial approximate root of the equation.

Step 4 : The next successive approximations to the root are given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$ and so on upto the desired accuracy by putting n=0, 1, 2, 3 ....

in general formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Useful deduction from the Newton Raphson Formula :

1. Iteration formula to find $\frac{1}{N}$ is $x_{n+1} = x_n(2 - Nx_n)$
2. Iteration formula to find $\sqrt{N}$ is $x_{n+1} = \frac{1}{2}(x_n + N/x_n)$
3. Iteration formula to find $\sqrt[k]{N}$ is $x_{n+1} = \frac{1}{K}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right]$
4. Iteration formula to find $\frac{1}{\sqrt{N}}$ is $x_{n+1} = \frac{1}{2}(x_n + \frac{1}{Nx_n})$

Proof : (1) Let x = $\frac{1}{N}$

$\Rightarrow N = \frac{1}{x}$

$\Rightarrow \frac{1}{x} - N = 0$

Taking f(x) = $\frac{1}{x} - N$, $f'(x) = -x^{-2}$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\left(\frac{1}{x_n} - N\right)}{-x_n^{-2}} = x_n + \left(\frac{1}{x_n} - N\right)x_n^2 = x_n + x_n - Nx_n^2$$

$= 2x_n - Nx_n^2$

$\Rightarrow x_{n+1} = x_n(2 - Nx_n)$

Proof : (2) Let x = $\sqrt{N} \Rightarrow x^2 - N = 0$

Take f(x) = x_2 - N, $f'(x) = 2x$

By Newton's formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{x_n^2 - N}{2x_n} = \frac{2x_n^2 - x_n^2 + N}{2x_n} = \frac{x_n^2 + N}{2x_n}$$

Hence, $x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$

Proof : (3) Let $x = \sqrt[k]{N} \Rightarrow x^k - N = 0$

Take $f(x) = x^k - N, \; f'(x) = kx^{k-1}$

By Newton's formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{x_n^k - N}{kx_n^{k-1}} = \frac{kx_n^k - x_n^k + N}{kx_n^{k-1}} = \frac{(k-1)x_n^k + N}{kx_n^{k-1}}$$

Hence, $x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right]$

Proof : 4 Let $x = \frac{1}{\sqrt{N}} \Rightarrow x^2 - \frac{1}{N} = 0$

Take f(x) = $x^2 - \frac{1}{N}, \quad f'(x) = 2x$

By Newton's formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{(x_n^2 - \frac{1}{N})}{2x_n} = \frac{2x_n^2 - x_n^2 + \frac{1}{N}}{2x_n} = \frac{x_n^2 + \frac{1}{N}}{2x_n}$$

Hence, $x_{n+1} = \frac{1}{2}\left[x_n + \frac{1}{Nx_n}\right]$

Example :

Evaluate $\frac{1}{31}$ by Newton's iterative method(Correct to four decimal places)

Proof : Let x = $\frac{1}{31}$

$\Rightarrow \quad \frac{1}{x} = 31$

$\Rightarrow \quad \frac{1}{x} - 31 = 0$

Taking f(x) = $\frac{1}{x} - 31$ we have $f'(x) = \frac{-1}{x^2} = -x^{-2}$

Then Newton's formula given

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\left(\frac{1}{x_n} - 31\right)}{-x_n^{-2}}$$

$\Rightarrow \quad x_{n+1} = x_n + \left(\frac{1}{x_n} - 31\right)x_n^2$

$\Rightarrow \quad x_{n+1} = x_n + x_n - 31x_n^2 = x_n(2 - 31x_n)$

So, $x_{n+1} = x_n(2 - 31x_n)$

Since an approximate value of $\frac{1}{31} = 0.03$

Let x_0 = 0.03

$x_1 = x_0(2-31x_0) = 0.03(2 - 31x0.03) = 0.0321$

$x_2 = x_1(2 - 31x_1) = 0.0321(2 - 31x0.03226) = 0.03226$

$x_3 = x_2(2-31. x_2)$ = 0.03226 (2-31x0.03226) = 0.03226

Since $x_2 = x_3$ upto 4 decimal places we have $\frac{1}{31} = 0.03226$

Example

By applying Newton's iterative method twice, Find a real root near 2 of the equation x_4-12x+7=0

Solution Given x_4 - 12x +7=0

$\Rightarrow$ f(x) = x_4-12x+7

Step 1 $f'(x)$ =4x_3 -12 = 4(x_3 - 3)

Step 2 x 0 1 2 3

f(x)	7	-4	-1	52

One root lies in [0, 1] and another root lies in [2, 3]

Taking $x_0 = 2$

$f(x_0) = 2_4 - 12 \times 2 + 7 = -1$

$$f'(x_0) = 4(2^3 - 3) = 20$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{-1}{20} = 2 + \frac{1}{20} = 2.05$$

$f(x_1) = (2.05)_4 - 12(2.05) + 7 = 0.06101$

$$f'(x_1) = 4[(2.05)^3 - 3] = 22.46052$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.05 - \frac{0.06101}{22.46052} = 2.04728$$

So, the real root by Appling Newton's iterative method twice is 2.048

Important long questions with solution.

1. Find the root of following equation using the bisection method correct up to two decimal placex_3 – 5x + 1 = 0 which lies between 2 and 3.

Solution : Given $f(x) = x_3 - 5x + 1$

and the root lies between 2 and 3.

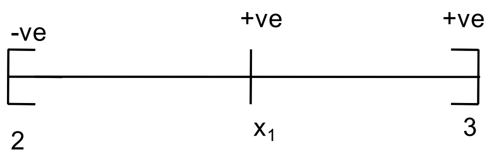
Now $f(2) = 2_3 - 5.2 + 1 = -1 < 0$

$f(3) = 3_3 - 5.3 + 1 = 13 > 0$

f(2).f(3) < 0, so root lies between 2 and 3,

Step-1

$$x_1 = \frac{2+3}{2} = 2.5$$

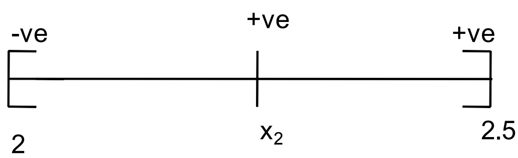


f (2.5) = 4.125>0

$\therefore$ f(2.5) . f(2) < 0, so root lies between 2 and 2.5.

Step-2

$$x_2 = \frac{2+2.5}{2} = 2.25$$

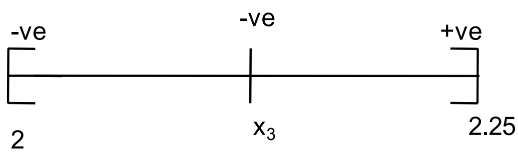


f(2.25) = 1.140625 > 0

$\therefore$ f(2) . f(2.25) < 0, so root lies between 2 and 2.25.

Step-3

$$x_3 = \frac{2+2.5}{2} = 2.125$$

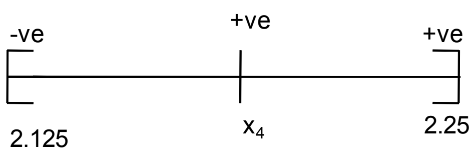


f(2.125) = – 0.02929688 < 0

f(2.125).f(2.25)<0, so root lies between 2.125 and 2.25.

Step- 4

$$x_4 = \frac{2.125+2.25}{2} = 2.1875$$

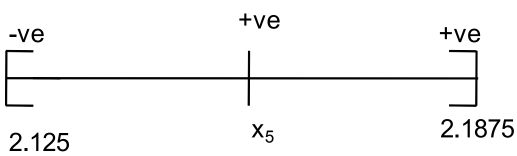


f(2.1875) = 0.5300293 > 0

f(2.125).f(2.1875)<0, so root lies between 2.125 and 2.1875.

Step-5

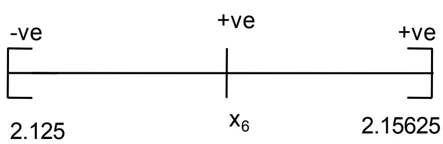
$$x_5 = \frac{2.125+2.1875}{2} = 2.15625$$



f(2.15625) = 0.2440490723 > 0

f(2.125).f(2.15625)<0, so that the root lies between 2.125 and 2.15625.

Step-6



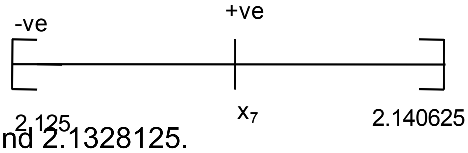
$$\therefore \quad x_6 = \frac{2.125 + 2.15625}{2} = 2.140625$$

f(2.140625) = 0.1058082581 > 0

f(2.125).f(2.140625)<0, so that the root lies between 2.125 and 2.140625.

Step-7

$$\therefore \quad x_7 = \frac{2.125 + 2.140625}{2} = 2.1328125$$

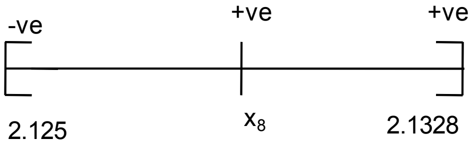


f(2.1328125) = 0.0378651619 > 0

f(2.125).f(2.1328125)<0, so that the root lies between 2.125 and 2.1328125.

Step-8

$$\therefore \quad x_8 = \frac{2.125 + 2.1328125}{2} = 2.\underline{\overline{12}}890625$$



f(2.12890625) = 0.004186689854 > 0

$f(2.\underline{12}5).f(2.\underline{12}890625)$ <0, so the root lies between 2.125 and 2.12890625.

$$\therefore \quad x_9 = \frac{2.125 + 2.12890625}{2}$$

$= 2.\underline{12}6953125$

$\therefore$ Hence our root is correct upto 2 decimal places. i.e. x = 2.12

Q. 2 Using Newton's Raphson formula, find the root of the equation correct upto three decimal places $xe_x - 2 = 0$

Ans. :

Given $xe_x - 2 = 0$

$\Rightarrow f(x) = xe_x - 2$

$f'(x) = e^x + xe^x$

$= (1 + x)e_x$.

$\therefore \quad f(x_n) = x_n e_{xn} - 2$

and $f'(x_n) = (1 + x_n)e^{xn}$

x	= 0	1
f(x)	= –2	0.718

$\therefore$ The root will lie between 0 and 1.

Taking $x_0 = \frac{0+1}{2} = 0.5$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n e^{xn} - 2}{(1 + x_n)e^{xn}}$$

$$= \frac{x_n e^{xn} + x_n^2 e^{xn} - x_n e^{xn} + 2}{(1 + x_n)e^{xn}}$$

$$\Rightarrow x_{n+1} = \frac{2 + x_n^2 e^{xn}}{(1 + x_n)e^{xn}}$$

Now $x_0 = 0.5$

$$\therefore \quad x_1 = \frac{2 + x_0^2 e^{xo}}{(1 + x_0)e^{xo}} = 0.975374213$$

$$x_2 = \frac{2 + x_1^2 e^{x_1}}{(1 + x_1)e^{x_1}} = 0.8633591061$$

$$x_3 = \frac{x_2^2 e^{x_1} + 2}{(1 + x_2)e^{x_2}} = 0.8526939231$$

$$x_4 = \frac{x_3^2 e^{x_{31}} + 2}{(1 + x_3)e^{x_3}} = 0.852605508$$

∴ Hence, our problem is correct upto four decimal places, i.e. x = 0.8526

Q. 3 Evaluate $\sqrt[3]{41}$ to four decimal places by Newton's interative methods.

Ans. :

Let x = $\sqrt[3]{41}$

$\Rightarrow x^3 = 41$

$\Rightarrow x^3 - 41 = 0$

$\therefore f(x) = x^3 - 41$, $\therefore f(x_n) = x_n^3 - 41$

$f'(x) = 3x^2$, $f'(x_n) = 3x_n^2$

x =	1	2	3	4
f(x) =	–40	–33	–14	17

f(3).f(4)<0, so that the root lies between 3 and 4.

Let $x_0 = \frac{3+4}{2} = 3.5$

The next approximation will be obtained by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - 41}{3x_n^2} = \frac{3x_n^3 - x_n^3 + 41}{3x_n^2}$$

$$\Rightarrow x_{n+1} = \frac{2x_n^3 + 41}{3x_n^2}$$

As x_0 = 3.5

$$x_1 = \frac{2.x_0^3 + 41}{3x_0^2} = 3.448979592$$

$$x_2 = \frac{2x_1^3 + 41}{3x_1^2} = 3.448217909$$

$$x_3 = \frac{2.x_2^3 + 41}{3x_2^2} = 3.44821724$$

∴ Hence our problem is correct upto four decimal places, i.e x = 3.4482.

Q. 4 By Newton-Raphson method, find real root of the equation log x – cos x = 0

Ans. :

Given that log x – cos x = 0

$\Rightarrow f(x) = \log x - \cos x$

$$f'(x) = \frac{1}{x} + \sin x = \frac{1 + x\sin x}{x}$$

$\therefore f(x_n) = \log x_n - \cos x_n$

$$f'(x_n) = \frac{1 + x_n \sin x_n}{x_n}$$

f(x) = log x – cos x

f(1) = log (1) – cos (1) = –0.54<0

f(2) = log (2) – cos (2) = 0.717>0

$\therefore$ f(1).f(2) < 0

root lies between 1 & 2

Choose x_0 = 1.5

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\log x_n - \cos x_n}{\frac{1}{x_n} + \sin x_n}$$

$$= \frac{x_n(\frac{1}{x_n} + \sin x_n) - (\log x_n - \cos x_n)}{(\frac{1}{x_n} + \sin x_n)}$$

$$= \frac{1 + x_n \sin x_n - \log x_n + \cos x_n}{\frac{1}{x_n} + \sin x_n}$$

For n = 0

$$x_1 = \frac{1 + x_0 \sin x_0 - \log x_0 + \cos x_0}{\frac{1}{x_0} + \sin x_0}$$

$$= \frac{1 + (1.5)\sin(1.5) - \log(1.5) + \cos(1.5)}{\frac{1}{1.5} + \sin 1.5}$$

= 1.436692414

For n = 1

$$x_2 = \frac{1 + x_1 \sin x_1 - \log x_1 + \cos x_1}{\frac{1}{x_1} + \sin x_1} = 1.42266718$$

For n = 2

$$x_3 = \frac{1 + x_2 \sin x_2 - \log x_2 + \cos x_2}{\frac{1}{x_2} + \sin x_2} = 1.419407466$$

For n = 3

$$x_4 = \frac{1 + x_3 \sin x_3 - \log x_3 + \cos x_3}{\frac{1}{x_3} + \sin x_3}$$

= 1.418642097

For n = 4

$$x_5 = \frac{1 + x_4 \sin x_4 - \log x_4 + \cos x_4}{\frac{1}{x4} + \sin x_4} = 1.418474302$$

My problem is correct up to 3 decimal i.e. 1.418

Q. 5 Using besection method, find the real root of the equation x + log x = 3.375, correct up to three decimal places.

Solution :

x + log x = 3.375

$\Rightarrow$ x + logx – 3.375 = 0

$\therefore$ f(x) = x + logx – 3.375

log 0 = ∞ , log 1 = 0, log 2 = .3010, log 3 = .4773

f(1) = 1 + log (1) – 3.375 = –2.375 < 0

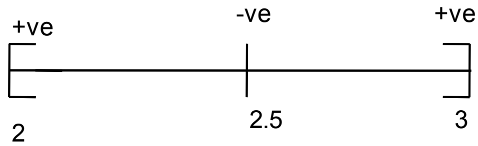
f(2) = 2 + log (2) – 3.375 = –1.0739<0

f(3)= 3+log(3) = 3.375 = 0.1021>0 f(2).f(3)<0

So root lies between 2 & 3

Step-1

$$x_1 = \frac{2+3}{2} = 2.5$$



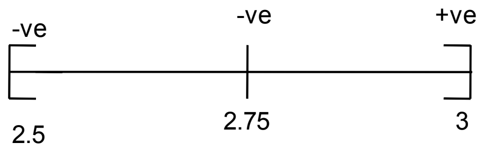
f(2.5) = 2.5 + log (2.5) – 3.375 = –0.477<0

f(2.5).f(3)<0

$\therefore$ root lies between 2.5 & 3

Step-2

$$x_2 = \frac{2.5+3}{2} = 2.75$$



f(2.75) = 2.75 + log (2.75) – 3.375 = –0.185 < 0

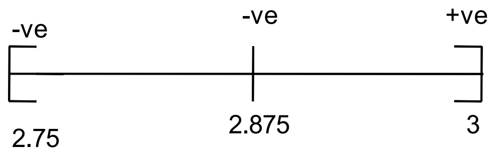
f(2.75).f(3)<0

root lies between 2-75 & 3.

Step-3

$$x_3 = \frac{2.75+3}{2} = 2.875$$

f(2.875) =2.875 + log (2.875) – 3.375 = –0.041<0



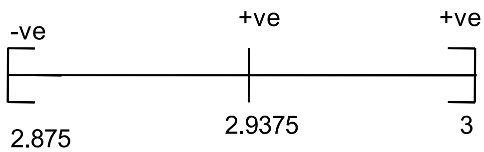
f(2.875).f(3)<0

root lies between 2.875 and 3

Step-4

$$x_4 = \frac{2.875+3}{2} = 2.9375$$

f(2.9375) = 2.9375 + log (2.9375) – 3.375= 0.03>0

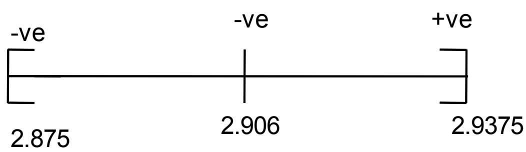


f(2.9375).f(2.875)<0

root lies between 2.875 & 2.9375

Step-5

$$x_5 = \frac{2.875+2.9375}{2} = 2.90625$$

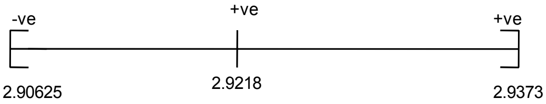


f(2.90625)=–5.417 x 10_{-3}<0

$\therefore$ f(2.90625).f(2.9375)<0

root lies between 2.90625 & 2.9375

Step-6



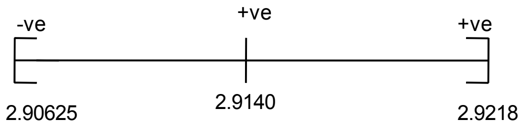
$$x_6 = \frac{2.90625 + 2.9325}{2} = 2.921875$$

f(2.921875) = 2.921875 + log (2.921875) – 3.375 = 0.0125>0

∴ f(2.921875). f(2.90625)<0

root lies between 2.921875 & 2.90625

Step-7

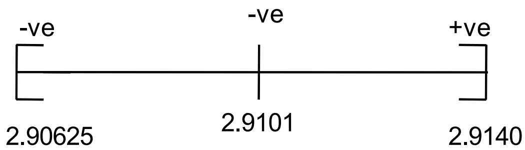


$$x_7 = \frac{2.90625 + 2.921875}{2} = 2.9140625$$

f(2.9140625)=3.561 x 10_{-3}>0

root lies between 2.9140625 & 2.90625

Step-8



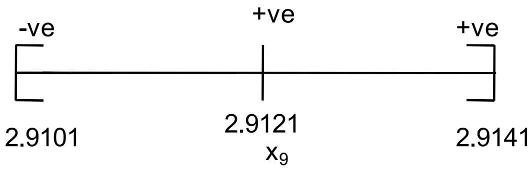
$$x_8 = \frac{2.9140625 + 2.90625}{2} = 2.91015625$$

f(2.91015625) = 2.91015625 + log (2.91015625) – 3.375

= – 8.66 x 10_{-4}<0

Step-9

$$x_9 = \frac{2.91015625 + 2.9140625}{2} = 2.912109325$$



f(2.912109325)

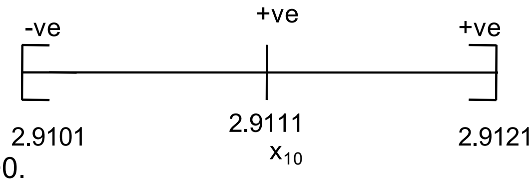
= 2.912109325+log(2.912109325 - 3.375

= 1.317 x 10_{-3} > 0

f(2.912109325) x f(2.91015625) < 0

So root lies between 2.912109325 and 2.91015625

Step - 10



$$x_{10} = \frac{2.912109375 + 2.91015625}{2} = 2.911132813$$

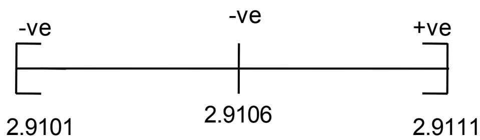
f(2.911132813)

=2.911132813 + log(2.911132813)-3.375 = 1.948 x 10_{-4}>0.

f(2.911132813). f(2.91015625) < 0

So root lies between 2.911132813 & 2.91015625

Step - 11



$$x_{11} = \frac{2.911132813 + 2.91015625}{2} = 2.910644532$$

f(2.910644532)= 2.910644532 + log(2.910644532)-3.375 = -3.662 x 10_{-4} <0

f(2.910644532).f(2.91132813) <0

∴ So the root is correct up to 2-decimal places i.e. 2.91

Chapter - 7
Finite Difference and Interpolation

7.1 We can find value of y for any value of x, when the function is defined but when a function y = f(x) is not defined, for some set of values of x, values of y are given

x	x_0	x_1	x_2......................	x_n
y	y_0	y_1	y_2......................	y_n

then the process of finding the values of y corresponding to any value of $x = x_i$ is called interpolation.

Thus **Interpolation** is the process of finding the value of the function for intermediate value of dependant variable x when the function is not given.

The process of finding value of the functions for some value of dependant variable outside the range is known as **extrapolation.**

The study of interpolation is based on the concept of differences of a function.

We have two types of difference in our syllabus.

They are (1) Forward differences

(2) Backward differences

Suppose a function y = f(x) is tabulated for equal spaced values of $x = x_0, x_0+h, x_0+2h, ... x_0+nh$ giving $y = y_0, y_1, y_2, ...y_n$.

1. Forward difference : Then the differences $y_1 - y_0, y_2-y_1.........y_n-y_{n-1}$ are denoted as $y_0, y_1, ...y_{n-1}$ are 1st forward differences. Where is the forward difference operator.

i.e. $\Delta y_n = y_{n+1} - y_n$

$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0)$

$= \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$

$= y_2 - y_1 - y_1 + y_0$

$\Delta^2 y_0 = y_2 - 2y_1 + y_0$

Proceeding likewise, $\Delta^n y_0 = y_n - n_{c_1} y_{n-1} + n_{c_2} y_{n-2} + ... + (-1)^n y_0$

Forward difference Table -

Value of x	value of y	1st diff.	2nd diff.	3rd diff	4th diff.	5th diff.
x_0	y_0					
		Δy_0				
$x_1=x_0+h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_2=x_0+2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_3=x_0+3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_4=x_0+4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_5=x_0+5h$	y_5					

In the above difference table, x is called the argument, y is the function or the entry, y_0 is the 1st entry and the leading term $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0, \Delta^5 y_0$ are called the leading differences.

2. Backward differences

The differences y_1-y_0, y_2-y_1, ...y_n-y_{n-1} denoted as ∇y_1, ∇y_2.......... ∇y_n respctively are called 1st backward differences. Where ∇ is the backward difference operator.

i.e. $\nabla y_n = y_n - y_{n-1}$

∇y_0 is not defined.

Backward difference table-

Value of x	value of y	1st diff.	2nd diff.	3rd diff.	4th diff.	5th diff.
x_0	y_0					
		∇y_1				
$x_1=x_0+h$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
$x_2=x_0+2h$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
$x_3=x_0+3h$	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
$x_4=x_0+4h$	y_4		$\nabla^2 y_5$			
		∇y_5				
$x_5=x_0+5h$	y_5					

In the above difference table, x is called the argument, y the function or the entry, y_0 the 1st entry is the leading term

Note : $\Delta^n y_0 = \nabla^n y_n$

Example :

Evaluate (i) Δ ($\tan_{-1}x$) (ii) $\Delta^2 e^x$ (iii) $\nabla \cos x$ taking h = 1

Sol. i) $f(x) = \tan_{-1}x$

$f(x+h) = \tan_{-1}(x+h)$

$\Delta f(x) = f(x+h)-f(x)$

$\Delta \tan_{-1}x = \tan_{-1}(x+h)-\tan_{-1}x$

$= \tan^{-1}\left(\frac{x+h-x}{1+x(x+h)}\right)$

$= \tan^{-1}\left(\frac{h}{1+hx+x^2}\right)$

Where h is the interval of difference

i.e. $h = x_1-x_0= x_2 - x_1=$ ....

if h = 1 i.e interval of difference being unity.

$$\Delta \tan^{-1} x = \tan^{-1}\left(\frac{1}{1+x+x^2}\right)$$

ii) $f(x) = e_x$

$f(x+h) = e_{x+h}$

$\Delta f(x) = f(x+h) - f(x)$

$\Delta e_x = e_{x+h} - e_x = e_x . e_h - e_x$

$= e_x(e_h - 1)$

If $h = 1$, $\Delta e_x = e_x(e-1)$

$\Delta_2 e_x = \Delta(\Delta e_x)$

$= \Delta[e^x(e^h - 1)] = (e^h - 1)\Delta(e^x)$

$= (e^h - 1)[e^x(e^h - 1)]$

$= (e_h - 1)_2 e_x$

iii) $f(x) = \cos x$

$f(x-h) = \cos(x-h)$

$\nabla f(x) = f(x) - f(x-h)$

$= \cos x - \cos(x-h)$

$= 2\sin\left(\frac{x+x-h}{2}\right).\sin\left(\frac{x-h+x}{2}\right)$

$= 2\sin\left(\frac{2x-h}{2}\right).\sin\left(\frac{-h}{2}\right)$

$= 2\sin\left(x - \frac{h}{2}\right).\left(-\sin\frac{h}{2}\right)$

when $h=1$, $\nabla \cos x = -2\sin\left(x - \frac{1}{2}\right).\sin\frac{1}{2}$

Difference of a polynomial

We know that the expression of the form $f(x) = a_0 x_n + a_1 x_{n-1} + ... + a_{n-1}x + a_n$ where a_n's are constant ($a_0 \neq 0$) and n is a positive integer is called a polynomial in x of degree n.

Theorem :

The 1st difference is a polynomial of degree n is of degrees n-1, the 2nd difference is of degree n -2, and the n_{th} difference is constant. While higher difference are equal to zero.

The converse of the theorem is also true which stated that if n_{th} difference of a function tabulated at equally space intervals are constant, the function is a polynomial of degree n.

Example : Form the suceessive forward difference of ax_3, the interval being h

Solution : Here $y = f(x) = ax_3$

We know that $\Delta y_0 = y_1 - y_0 = f(x+h) - f(x)$

$\Delta(ax^3) = a(x+h)^3 - ax^3$

$= a(x^3 + 3x^2h + 3xh^2 + h^3) - ax^3$

$= a(3x_2h + 3xh_2 + h_3)$

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$

$\Delta^2(ax^3)$= $a[3(x+h)_2h+3(x+h)h_2 +h_3] - a[3x_2h+3xh_2 +h_3]$

$= a\{3x_2h+6xh_2+3h_3+3xh_2+3h_3+h_3 - 3x_2h - 3xh_2-h_3\} = a\{6xh_2|+6h_3\}$

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$

$\Delta^3(ax^3)$ = $a\{6(x+h)h_2+6h_3\}-a\{6xh_2+6h_3\}$

$= a\{6xh_2+6h_3+6h_3-6xh_2-6h_3\}$

$= 6a\ h_3$ = constant.

$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$

$\Delta^3(ax^3) = 6ah^3 - 6ah^3 = 0$

Here it shows that the third differences of a polynomial of third degree is constant & the higher difference are zero.

7.2 Shift operator (E)

Shift operator is denoted as E is defined as E f(x) = f(x+ h)

i.e. $Ey_0 = y_1$ $\quad E_2y_0= E(Ey_0)$

$Ey_1 = y_2$ $\quad = E(y_1)$

... $\quad =y_2$

...

$Ey_n = y_{n+1}$ $\quad E_ny_0 = y_n$

Inverse shift Operator (E_{-1}) :

Inverse shift operator E_{-1} is the operation of decreasing the argument x by h. i.e. $E_{-1}f(x) = f(x-h)$

$E_{-1}y_n = y_{n-1}$

$E_{-1}y_{n-1}= y_{n-2}$ ....

$E_{-1}y_2 = y_1$ $\quad E_{-1}y_1 = y_0$

Note : $E_{-1}\ y_0$ is not defined.

Relation between Δ, ∇ **and E :**

i) $\Delta= E-1$ ii) $\nabla =1-E_{-1}$ iii) $\Delta= E\nabla = \nabla E$

Proof :

i) $\Delta y_n=y_{n+1}-y_n$

$= Ey_n - y_n = (E-1)y_n$

$\Rightarrow \Delta=E-1$

ii) $\nabla y_n = y_n-y_{n-1}$

$= y_n - E_{-1}y_n$

$= (1-E_{-1})y_n$

$\Rightarrow \nabla =1-E_{-1}$

iii) $\Delta y_n=y_{n+1}-y_n$

$= Ey_n - Ey_{n-1} = E(y_n - y_{n-1})$

$= E \nabla y_n$

$\Rightarrow \Delta = E\nabla$ ....(i)

$\nabla E y_n = \nabla (y_{n+1})$

$= y_{n+1} - y_n$

$= \Delta y_n$

$\Rightarrow \Delta = \nabla E$ .... (ii)

From (i) & (ii) $\Delta = \nabla E = E\nabla$

Example : Estimate the missing term in the following table

x	0	1	2	3	4
f(x)	1	3	9	-	81

Solution :

Let the missing term y_3

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		2			
1	3		4		
				$y_3 - 19$	
		6			$124 - 4y_3$
2	9		$y_3 - 15$		
		$y_3 - 9$		$105 - 3y_3$	
3	y_3		$81 - 2y_3 + 9$		
		$81 - y_3$			
4	81				

As only four entries y_0, y_1, y_2, y_3 are given, the function y can be represented by a third degree polynomial, here 4_{th} order difference become zero i.e.

$124 - 4y_3 = 0 \Rightarrow y_3 = 31$

Hence, missing term is 31.

Example :

Estimate the missing term in the following table.

x	0	1	2	3	4	5	6
y	5	11	22	40	-	140	-

Solution :

Let the missing term by y_4 & y_6.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	5					
		6				
1	11		5			
		11		2		
2	22		7		y_4-67	
		18		y_4-65		$370-5y_4$
3	40		y_4-58		$303-4y_4$	
		y_4-40		$238-3y_4$		y_6+10y_4-1001
4	y_4		$180-2y_4$		y_6+6y_4-698	
		$140-y_4$		y_6+3y_4-460		
5	140		y_6+y_4-280			
		y_6-140				
6	y_6					

As only four entries y_0, y_1, y_2, y_4, y_5 are given The function y can be represented by a 4th degree polynomial & hence 5th difference become Zero i.e.

$370-5y_4 = 0$ and $y_6+10y_4-1001 = 0$

$\Rightarrow y_4=74$ $\Rightarrow$ $y_6+10 \times 74-1001=0$

$\Rightarrow$ $y_6+740-1001=0$

$\Rightarrow$ $y_6 = 261$

7.3 Newton's Interpolation formulae :

It is of two types

i) Newton forward interpolation

ii) Newton backward interpolation

i) Newton's forward interpolation formula :

Let the function $y=f(x)$ have values $y_0, y_1, y_2 \ldots\ldots y_n$ corresponding to $x = x_0, x_0+h, x_0+2h, \ldots\ldots x_0+nh$.

Suppose we have to find value of y for some intermediate value of $x = x_p$, where $x_p = x_0+ph$

$f(x_p) = f(x_0+Ph)$

$\Rightarrow \quad y_p = E^P f(x_0)$

$= (1+\Delta)^p y_0 \qquad (E = 1+\Delta)$

$$= \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\right]y_0 \qquad \text{(Expanding Binomially)}$$

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

Where $P = \dfrac{x_p - x_0}{h}$

Note : This formula is used for interpolating the value of y near the beginning of a set of tabulated values and extrapolating values of y a little backward of y_0

ii) Newton's Backward interpolation formula :

Let the function y = f(x) have values y_0, y_1.... y_n corresponding the value of x = x_0, x_0+h, x_0+2h,....x_0+nh respectively.

Suppose we have to find y for some intermediate value of x = x_p,

Where $x_p = x_n + ph$

$$f(x_p) = f(x_n+ph)$$

$$= E^P f(x_n)$$

$$y_P = (1-\Delta)_{-p} y_n \qquad E_{-1}=(1-\Delta) \Rightarrow E = (1-\Delta)_{-1}$$

$$= \left[1 + p\nabla + \frac{p(P+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + ...\right] y_n$$

$$y_p = y_n + p\nabla y_n + \frac{p(P+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + ...$$

Where $p = \dfrac{x_p - x_n}{h}$

Note : This formula is used for interpolating value of y near the end of a set of tabulated values or extrapolating values of y a little ahead of y_n

Example -1

The table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earth's surface :

x = height :	100	150	200	250	300	350	400
y = distance :	10.63	13.3	15.04	16.81	18.42	19.90	21.27

Find the values of y when (i) x = 218 ft ii) 410ft

Sol. The difference table is

x	**y**	**Δ**	Δ_2	Δ_3	Δ_4
100	10.63				
		2.40			
150	13.03		- 0.39		
		2.01		0.15	
200	**15.04**		- 0.24		- 0.07
		1.77		0.08	
250	16.81		**- 0.16**		- 0.05
		1.61		**0.03**	
300	18.42		- 0.13		**- 0.01**
		1.48		**0.02**	
350	19.90		**- 0.11**		
		1.37			
400	**21.27**				

i) If we take x_0 = 200, then y_0 = 15.04, y_0 =1.77, ${}_2y_0$ = -0.16, ${}_3y_0$ = -0.03, ${}_4y_0$ = -0.01

Since x = 218 and h = 50, $\therefore p = \dfrac{x - x_0}{h} = \dfrac{18}{50} = 0.36$

∴ Using Newton's forward interpolation formula,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36(-0.64)}{2}(-0.18) + \frac{0.36(-0.64)(-1.64)}{6}(0.03) + \frac{0.36(-0.64)(-1.64)(-2.64)}{24}(-0.01)$$

= 15.696 i.e. 15.7 nautical miles

ii) Since x = 410 is near the end of the table, we use Newton's backward interpolation formula.

∴ Taking $x_n = 400, p = \frac{x - x_n}{h} = \frac{10}{50} = 0.2$

Using the line of backward differences

$y_n = 21.27,\ \nabla y_n = 1.37, \nabla^2 y_n = -0.11,\ \nabla^3 y_n = 0.02$ etc.

By Newton's backward formula

$$y_p = y_n + p\nabla y_n + \frac{p(P+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$f(410) = 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2}(-0.11) + \frac{0.2(1.2)(2.2)}{6}(0.02) + \frac{0.2(1.2)(2.2)(3.2)}{24}(-0.01)$$

= 21.53 nautical miles.

Example : 2

From the following table, estimate the number of students who obtained marks between 40 and 45.

Marks	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

Sol. First we prepare the cumulative frequency table

Mark less than (x)	:	40	50	60	70	80
No. of students (y_x)	:	31	73	124	159	190

Now the difference table is

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	**31**				
		42			
50	73		**9**		
		51		**- 25**	
60	124		- 16		37
		35		12	
70	159		- 4		
		31			
80	190				

Taking $x_0 = 40$, $x=45$, $p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5$

No. of students with marks less than 45 i.e.

$$f(45) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$= 31 + 0.5x42 + \frac{0.5(-0.5)}{2}x9 + \frac{0.5(-0.5)(-1.5)}{6}x(-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24}x37$

= 47.87

The number of students with marks less than 40 is 31

Hence the number of students getting marks between 40 and 45 = 48-31 = 17

Example : 3

Find the cubic polynomial which takes the following values :

x	:	0	1	2	3
f(x)	:	1	2	1	10

Evaluate f(4)

Sol. The difference table is

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	**1**			
		1		
1	2		**-2**	
		-1		**12**
2	1		10	
		9		
3	10			

Taking x_0= 0, p = $\frac{x-0}{h} = x$

Using Newton's forward interpolation formula,

$f(x_p) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$

$f(x) = f(0) + \frac{x}{1}\Delta f(0) + \frac{x(x-1)}{1.2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 f(0)$

$= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$

$= 1 + x - x^2 + x + 2(x^2 - x)(x-2) = 1 + 2x - x^2 + 2(x^3 - 3x^2 + 2x) = 1 + 2x - x^2 + 2x^3 - 6x^2 + 4x$

$= 2x_3 - 7x_2 + 6x + 1$

$f(4) = 2.4_3 - 7.4_2 + 6.4 + 1$

= 2.64 - 7.16 + 24 +1

= 128 - 112 + 25 = 41

Which is the same value as that obtained by substituting x= 4 in the cubic polynomial abover

f(4) = 41

Example : 4

Find the number of men getting wage between Rs 10 and Rs 15 from the following data.

Wages in Rs	0-10	10-20	20-30	30-40
Frequency	9	30	35	42

Solution :

First we prepare the cummulative frequency table, as follows

Wages less then x	10	20	30	40

No of men	9	39	74	116

Now the difference table is

x	y	Δ	Δ^2	Δ^3
10	9			
		30		
20	39		5	
		35		2
30	74		7	
		42		
40	116			

We Shall find y_5 i..e number of men getting way less then 15,

Taking x_0 = 10, x = 15

$$P = \frac{x - x_0}{h} = \frac{15 - 10}{10} = \frac{5}{10} = 0.5$$

Using Newton forward intrpolation formula we get

$$y_p = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

$$= 9 + (.5)(30) + \frac{(.5)(.5-1)}{2} x5 + \frac{(.5(.5-1)(.5-2))}{6} x2$$

=9+15-0.625 + .125 = 23.5 = 24

Number of men getting wages between rs 10 and Rs 15 = 24-9=15

7.4 Lagrange's Interpolation formula for unequal intervals :

This formula is applicable when values of y = f(x) are $y_0, y_1 \ldots y_n$ for unequal values of $x=x_0, x_1, \ldots x_n$ are given, then

$$f(x) = \frac{(x-x_1)(x-x_2)....(x-x_n)}{(x_0-x_1)(x_0-x_2).....(x_0-x_n)} y_0 \frac{(x-x_0)(x-x_2)....(x-x_n)}{(x_1-x_0)(x_1-x_2).....(x_1-x_n)} y_1 + ... + \frac{(x-x_0)(x-x_1)....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1).....(x_n-x_{n-1})} y_n$$

is Lagrange's interpolation formula for unequal intervals.

Proof :

Let y=f(x) be a function which takes the values $(x_0, y_0), (x_1, y_1)....... (x_n, y_n)$.

Since there are (n+1) of values of x and y, we can represent f(x) by a polynomial in x of degree n.

Let the polynomial be of the form

$y=f(x) = a_0(x-x_1)(x-x_2)....(x-x_n)+a_1(x-x_0)(x-x_2)....(x-x_n)$

$+a_2(x-x_0)(x-x_1)(x-x_3)....(x-x_n)+......+ a_n(x-x_0)(x-x_1)(x-x_2).......(x-x_{n-1})$ ....(1)

Putting $x = x_0$, $y_0 = a_0 (x_0 - x_1)(x_0-x_2).....(x_0 - x_n)$

$$\Rightarrow \quad a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2).....(x_0-x_n)}$$

Simillarly putting $x = x_1$,

$y_1 = a_1(x_1-x_0)(x_1-x_2)(x_1-x_3).......(x_1-x_n)$

$$\Rightarrow \quad a_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2).......(x_1-x_n)}$$

Proceeding likewise, $a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1)......(x_n - x_{n-1})}$

Putting the values of a_0, a_1, a_2............a_n in the supposed polynomial (1)

$$f(x) = \frac{(x-x_1)(x-x_2).....(x-x_n)}{(x_0-x_1)(x_0-x_2).....(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2).....(x_1-x_n)}y_1 + ...$$

$$+ \frac{(x-x_0)(x-x_1).....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1).....(x_n-x_{n-1})}y_n$$

Example : Given the values

x	:	5	7	11	13	17
f(x)	:	150	392	1452	2366	5202

Evaluate f(9), using Lagrange's interpolation formula

Sol. Here $x_0 = 5$, $x_1 = 7$, $x_2 = 11$, $x_3 = 13$, $x_4 = 17$

$y_0=150$, $y_1=392$, $y_2 = 1452$, $y_3 = 2366$, $y_4 = 5202$

By Lagrange's Interpolation formula

$$F(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}y_3$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}y_4$$

Putting x =9 in above formula,

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}x150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}x392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}x1452 + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}x2366$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}x5202$$

$$= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810$$

Example :

Apply lagrange's method to find the value of x when f(x)=15 from the given data

x	5	6	9	11
f(x)	12	13	14	16

Solution :

Here $x_0=5$ $x_1=6$ $x_2=9$ $x_3=11$

$y_0=12$ $y_1=13$ $y_2=14$ $y_3=16$

Taking y = 15 and using the above results in lagrange's inverse interpolation formula.

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}.x_1$$

$$+\frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2+\frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}.x_3$$

$$=\frac{(15-13)(15-14)(15-16)}{(12-13)(12-14)(12-16)}.5+\frac{(15-12)(15-14)(15-16)}{(13-12)(13-14)(13-16)}.6$$

$$+\frac{(15-12)(15-13)(15-16)}{(14-12)(14-13)(14-16)}.9+\frac{(15-12)(15-13)(15-14)}{(16-12)(16-13)(16-14)}.11$$

$$=\frac{2x1(-1)}{(-1)(-2)(-4)}x5+\frac{3x1(-1)}{1(-1)(-3)}x6+\frac{3x2x(-1)}{2x1x(-2)}x9+\frac{3x2x1}{4x3x2}x11$$

$$=\frac{5}{4}-6+\frac{27}{2}+\frac{11}{4}=1.25-6+13.5+2.75=17.5.6=11.5$$

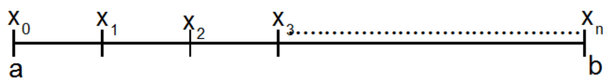
7.5 Numerical Integration :

The process of evaluating a definite integral from a set of tabulated values of the integrand f(x) is called numerical integration. This process when applied to a function of a single variable is known as quadrature.

7.5.1 Newton- cote's quadrature formula :

Let $I=\int_a^b f(x).dx$, where f(x) takes the values y_0, y_1, y_2 ....y_n for $x = x_0$, x_1.......x_n

Let's divide the interval (a,b) into n sub-intervals of width 'h' so that $x_0 = a$, $x_0+h=x_1$, $x_0+2h = x_2....x_0 + nh = x_n = b$, such as.



Then $I = \int_{x_0}^{x_0+nh} f(x).dx$

Put $x = x_0+rh$

$dx = hdr$

$x = x_0 \Rightarrow r = 0$

$x = x_0 +nh \Rightarrow r = n$

$$= h\int_0^n f(x_0+rh)dr$$

$$= h\int_0^n\left[y_0+r\Delta y_0+\frac{r(r-1)}{2!}\Delta^2 y_0+.......\right]dr$$

Integrating term by term

$$\int_{x_0}^{x_0+nh} f(x)dx = nh\left[y_0+\frac{n}{2}\Delta y_0+\frac{n(2n-3)}{12}\Delta^2 y_0+\frac{n(n-2)^2}{24}\Delta^3 y_0+.........\right] \quad .... (1)$$

Which is known as Newton-cote's quadrature formula.

7.5.2 For n=1 Trapizoidal Rule

Putting n=1 in equation (1) taking curve through (x_0, y_0) and (x_1, y_1)

$$\int_{x_0}^{x_0+h} f(x)dx = h\left[y_0+\frac{1}{2}\Delta y_0\right]$$

$$=\frac{h}{2}[y_0+y_1]$$

Simillarly $\int_{x_0+h}^{x_0+2h} f(x)dx = \frac{h}{2}\left[y_1 + \frac{1}{2}\Delta y_1\right] = \frac{h}{2}[y_1 + y_2]$

....

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)dx = \frac{h}{2}[y_{n-1} + y_n]$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})]$$

7.5.3 Simpson's rd Rule :

Putting n=2 equation (1) and taking curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) on parabola.

$$\int_{x_0}^{x_0+2h} f(x)dx = 2h(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0)$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Simillarly $\int_{x_0+2h}^{x_0+4h} f(x)dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$

....

$\int_{x_0+(n-2)h}^{x_0+nh} f(x)dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$ (n being even)

Adding, all these integrals, we have n is even

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2})]$$

Working procedure of Trapizoidal, and simpson's $\frac{1}{3}$ rd

Step - 1 Compare $\int_a^b f(x)dx$ with given problem

Step - 2 Choose a, b, f(x),

Take $x_0 = a$, $x_0+nh = b$, $h = \frac{b-a}{n}$

(n is the total no. of sub intervals, depend up on user)

Step - 3 Find x_0, x_1, x_2, ...x_n and corresponding y_0, y_1, y_2, y_3, ...y_n

Step - 4 For Trapizodal method

Set all values of y_0, y_1, y_2 ....y_n in

$$\frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})]$$

Step - 5 For Simpson $\frac{1}{3}$ th

Set all value of y_0, y_1, y_2 ....y_n in

$$= \int_a^b f(x)dx \; \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2})]$$

Example : 1

Evaluate $\int_0^6 \frac{1}{1+x^2}dx$ by using trapizodal method

Sol.

Step - 1 Here standard form is $\int_a^b f(x)dx$

given problem is $\int_0^6 \frac{1}{1+x^2}dx$

Step - 2 a = 0, b=6, f(x) = $\frac{1}{1+x^2}$

Take n=6. h = $\frac{b-a}{n} = \frac{6-0}{6} = 1$

x_0= 0, x_1 = x_0+ h x_4 = x_0+4h

= 0+1 = 1 = 0+4.1 = 4

x_2 = x_0+2h x_5 = x_0+5h

= 0+2.1 = 2 = 0+ 5.1 = 5

x_3 = x_0+ 3h x_6 = x_0 + 6h

= 0+3.1 = 3 = 0+6.1 = 6

Step - 3 For $x_0 = 0$ $y_0 = \frac{1}{1+x_0} = \frac{1}{1+0} = \frac{1}{1} = 1$

$x_1 = 1$ $y_1 = \frac{1}{1+x_1} = \frac{1}{1+1} = \frac{1}{2} = .5$

$x_2 = 2$ $y_2 = \frac{1}{1+2^2} = \frac{1}{5} = .20$

$x_3 = 3$ $y_3 = \frac{1}{1+3^2} = \frac{1}{10} = 0.1$

$x_4 = 4$ $y_4 = \frac{1}{1+4^2} = \frac{1}{17} = .0588$

$x_5 = 5$ $y_5 = \frac{1}{1+5^2} = \frac{1}{26} = .0385$

$x_6 = 6$ $y_6 = \frac{1}{1+6^2} = \frac{1}{37} = .027$

Step - 4 For Trapizodal

$\int_a^b f(x)dx = \frac{h}{2}[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)]$

$= \frac{1}{2}[(1+.027)+2(.5+.20+.1+.0588+.0385)]$

$= \frac{1}{2}[1.027)+2(0.8973)]$ = 1.4108

Step - 5 For Simpson's $\frac{1}{3}$ rd

$\int_a^b f(x)dx = \frac{h}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]$

$= \frac{1}{3}[(1+0.027)+4(0.5+0.1+0.0385)+2(0.2+0.0588)]$ = 1.3662

Long Questions with Answer

1. Using the Newton's forward Interpolation formula find the value as f(1.6) if.

x : 1 1.4 1.8 2.2

f(x)	:	3.49	4.82	5.96	6.5

Solution :

x	f(x)	Δ	Δ^2	Δ^3
1	3.49			
		1.33		
1.4	4.82		-0.19	
		1.14		-0.41
1.8	5.96		-0.6	
		0.54		
2.2	6.5			

X_p= 1.6

h = 1.4-1=0.4

x_0 = 1.4

$x_p = x_0+ph$

$\Rightarrow$ $ph = x_p - x_0$

$\Rightarrow$ $p = \dfrac{x_p - x_0}{h}$

$\therefore$ $p = \dfrac{1.6-1.4}{0.4} = 0.5$

Using Newton's forward interpolation formula we get

$Y_p = y_0 + p\Delta y_0 + \dfrac{p(p-l)}{2!}\Delta^2 y_0 + \dfrac{p(p-1)(p-2)}{31}\Delta^2 y_0.$

$f(1.6) = 4.82 + (0.5) \times (1.14) + \dfrac{(0.5)(0.5-1)}{2}(-0.6) = 5.465$

By using Newton forward interpolation formula we have got y = 5.465

2. Apply Newton's Backward formula to find polynomial of degree three which includes the following x, y pairs.

x	:	3	4	5	6
y	**:**	**6**	**24**	**60**	**120**

Solution : The difference table is

x	y	∇y	$\nabla_2 y$	$\nabla_3 y$
3	6			
		18		
4	24		18	
		36		6
5	60		24	
		60		
6	120			

$x_n = 6$, $y_n = 120$, $\nabla y_n = 60$, $\nabla_2 y_n = 24$, $\nabla_3 y_n = 6$

h=1, $p = \dfrac{x - x_n}{h} = \dfrac{x-6}{1} = x-6$

$y = y_n + p\nabla y_n + \dfrac{P(P+1)}{2!}\nabla^2 y_n + \dfrac{P(P+1)(P+2)}{3!}\nabla^3 y_n$

$= 120 + (x-6)60 + \dfrac{(x-6)(x-6+1)}{2} \times 24 + \dfrac{(x-6)(x-6+1)(x-6+2)}{6} \times 6$

= 120+60x-360+12(x-6)(x-5)+(x-6)(x-5)(x-4)

$= 120+60x-360+12x_2-132x+360+(x-6)[x_2-9x+20]$

$= 120+60x-360+12x_2-132x+360+x_3-9x_2+20x-6x_2+54-120$

$= x_3-3x_2+2x$

By using Newton Backward formula

we have obtained $f(x) = x_3-3x_2+2x$

3. Using lagrange's Interpolation find the value of y when x=10 if the following value x & y are given

x	**:**	**5**	**6**	**9**	**11**
y	**:**	**12**	**13**	**14**	**16**

Solution : Given Data

x	:	5	6	9	11
y	:	12	13	14	16
		0	1	2	3

Given x = 10

$$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x-x_3)} = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} = \frac{1}{6}$$

$$l_1 = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} = \frac{-1}{3}$$

$$l_2 = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} = \frac{5}{6}$$

$$l_3 = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_2-x_1)(x_3-x_2)} = \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} = \frac{1}{3}$$

$p(x) = l_0y_0+l_1y_1+l_2y_2+l_3y_3$

$$= \left(\frac{1}{6}x12\right)+\left(-\frac{1}{3}x13\right)+\left(\frac{5}{6}x14\right)+\left(\frac{1}{3}x16\right) = 14.66666667$$

4. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using trapezoidal rules and simpson's $\frac{1}{3}$ rd rule taking $h = \frac{1}{4}$ and compare the result with its actual value.

Solution :

Step - 1 Given $\int_0^1 \frac{dx}{1+x^2}$

The standard form is $\int_a^b f(x)dx$

Step - 2 Comparing we have

$f(x)= \frac{1}{1+x^2}$, a=0

b=1, $h= \frac{1}{4}$

Step - 3 $x_0 = 0=a$

$x_1 = x_0+h=0+\frac{1}{4} = 0.25$ $x_2 = x_0+2h=0+\frac{2}{4} = 0.5$

$x_3 = x_0 + 3h = \frac{3}{4} = 0.75$ $\qquad$ $x_4 = x_0 + 4h = \frac{4}{4} = 1 = b$

Step - 4 $\quad y_0 = \frac{1}{1+x^2}$

$$\Rightarrow y_0 = \frac{1}{1+x_0^2} = \frac{1}{1+0} = 1$$

$$y_1 = \frac{1}{1+x_1^2} = \frac{1}{1+(.25)^2} = \frac{1}{1.0625} = 0.941$$

$$y_2 = \frac{1}{1+x_2^2} = \frac{1}{1+(.5)^2} = \frac{1}{1+0.25} = \frac{1}{1.25} = 0.8$$

$$y_3 = \frac{1}{1+x_3^2} = \frac{1}{1+(0.75)^2} = \frac{1}{1+0.5625} = \frac{1}{1.5625} = 0.64$$

$$y_4 = \frac{1}{1+x_4^2} = \frac{1}{1+1^2} = \frac{1}{2} = 0.5$$

Step - 5 $\quad$ For trapezoidal rule

$$\int_0^b f(x)dx = \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{1}{\left(\frac{4}{2}\right)}\left[(1+0.5) + 2(0.941 + 0.8 + 0.64)\right]$$

$= 0.78275$

Step - 6 $\quad$ For Simpson's ⅓rd rule

$$\int_0^b f(x)dx = \frac{h}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2\right]$$

$$= \frac{\frac{1}{4}\left[(1+0.5) + 4(0.941 + 0.64) + 2x0.8\right]}{3}$$

$$= \frac{1}{12}\left[1.5 + 6.324 + 1.6\right] = .7853333$$

Step - 7 $\quad$ Now the actual value is

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x\Big]_0^1 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4} = 0.78571$$

Step - 8 $\quad$ Now comparing the actual value with Trapezoidal value is

Error = 0.78571 - 0.78275 = 0.00296

And comparing the actual value with Simpson's value is

Error = 0.78571 - 0.785331 = 0.00038

Step - 9 $\quad$ We observed that the result obtained in Simpson's rule is closer to actual value rather them Trapezoidal rule.

5. Find the value of $\int_1^2 \frac{dx}{x}$

a) by Simpson ⅓rd rule taking h = $\frac{1}{4}$

b) by Trapizodal rule taking h= $\frac{1}{4}$

Hence obtain the approximate value of Ln2.

Solution :

Here a=1, b=2, f(x)= $\frac{1}{x}$

choose n=10 $\Rightarrow h = \frac{b-a}{n} = \frac{2-1}{10} = \frac{1}{10} = .1$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x :	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
F(x) :	1	0.9090909091	0.833333333	0.7692307692	.7142857143	0.666666666	0.625	0.5882352941	0.55555555	0.5263157845	0.5

a) By using simpson $\frac{1}{3}$ rd

$$\int_a^b f(x).dx = \frac{h}{3}\left[\{f(x_0)+f(x_{10})\} + 2\{f(x_2)+f(x_4)+f(x_6)+f(x_8)\} + 4\{f(x_1)+f(x_3)+f(x_5)+f(x_7)\}\right]$$

$= \frac{0.1}{3}$ {(1 + 05) +2 {(0.83333333 + 07142857143 + 0.625 + .555555555) + 4(0.9090909091 + .7692307692 + 0.666666666 + .588235294 + .5263157845)} = .6931622307

$$\int_1^2 \frac{dx}{x} = [Lnx]_1^2 = Ln2 - Ln1 = Ln2 = .6931471806$$

b) By using Trapizodal

$$\int_a^b f(x).dx = \frac{h}{2}\left[\{f(x_0)+f(x_{10})\} + 2[f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)+f(x_6)+f(x_7) + f(x_8)+f(x_9)]\right]$$

=½ [(1+.5)+2[.9090909091+.8333333333 + .7692307692 + .7142857143 + .666666666 +.625 + .5882352941 + 0.555555555 + .5263157845]] = .69377141019